lec01

2025-03-17

Teoria dos tipos

Formalização de matemática

Fundamentos de matemática

Teoria dos conjuntos

Teoria das categorias

Matemática construtiva

Teoria das demonstrações

Lógica matemática (FOL, ...)

Teoria dos grupos, aneis, ...

Teoria dos inteiros, reais, ...

Geometria euclideana

λ-calculus

Programação funcional

Proof assistants

Theorem provers

Static-vs-dynamic

axiomas-vs-leis

Zermelo - Euclides

Representando números como conjuntos

Zermelo	von Neumann
$0 \equiv \{\} \equiv \emptyset$	$\{\}$
$1 \equiv \{\{\}\}$	$\{0\}$
$2 \equiv \{\{\{\}\}\}$	$\{0,1\}$
$\vdots$	$\vdots$

$$n \mid m \overset{\text{def}}{\iff} (\exists k \in \mathbb{Z})[n \cdot k = m]$$

$$A \subseteq B \overset{\text{sug}}{\iff} (\forall x)[x \in A \Rightarrow x \in B]$$

$0 \in 2$?

$3 \subseteq 5$? — essas perguntas deveriam mesmo ser perguntáveis?

OK sobre os $0, 1, 2, \ldots$ mas cadê o $\mathbb{N} = \{0, 1, 2, \ldots\}$?

Representando tuplas como conjuntos

$\langle 7, 2 \rangle \overset{imp}{\equiv} \{ 7, 2 \}$

$\langle 7, 2 \rangle \overset{imp}{\equiv} \{ \{7\}, \{7,2\} \}$

$A \times B \overset{imp}{\equiv} ?$

(TUP1) $\langle x, y \rangle = \langle x', y' \rangle \iff x = x' \ \& \ y = y'$

(TUP2) $\{ \langle a, b \rangle \mid a \in A, \ b \in B \}$ deve ser um conjunto

Sistemas Hilbert

? : $\alpha \supset (\beta \supset \alpha)$

? : $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

? : $(\neg\alpha \supset \neg\beta) \supset (\beta \supset \alpha)$

Como inferir: (Modus Ponens)

Tendo $\alpha \supset \beta$ e α inferir β

$\vdash A \to A$

Gentzen

Natural deduction

Sequent calculus

Natural deduction

$$\frac{A \text{ true} \qquad B \text{ true}}{A \& B \text{ true}} \ \&\text{-I} \qquad \frac{A \& B \text{ true}}{A \text{ true}} \ \&\text{-E}_1 \qquad \frac{A \& B \text{ true}}{B \text{ true}} \ \&\text{-E}_2$$

$$\frac{A \text{ prop} \qquad B \text{ prop}}{A \& B \text{ prop}}$$

$$\left.\frac{\dfrac{A \& B}{B} \qquad \dfrac{A \& B}{A}}{B \& A}\right\} \text{ comutatividade de } (\&)$$

$(\in)$ vs $(:)$, dynamic vs static, ...

```
x = ...
if (x é um int):
    ~~~
else:
    ~~~
```

compila

```
int x;
if ( o inteiro x ):
    ~~~
else:
    ~~~
```

não compila

sets

types

escrevemos $a \in A$

escrevemos $a : A$

Set theory ops/construções:

$\emptyset$, $\{_,_\}$, $\bigcup_$, $\mathcal{P}_$, $\bigcap_$

Type theory ops/construções:

(outras construções)

Internal ops vs external constructions

lec04

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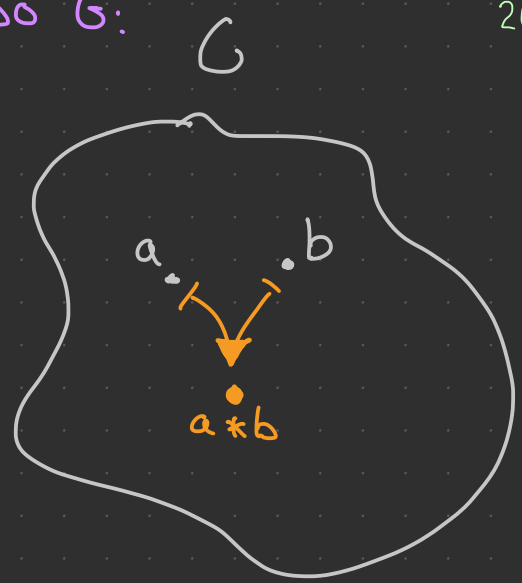
ops internas no grupo G:

$$(G\,;\ \mathrm{op},\ \mathrm{inv},\ \mathrm{id})$$

$$\mathrm{op} : G \times G \to G$$

$$\mathrm{id} : G$$

$$\mathrm{inv} : G \to G$$



ops externas ("constructions")

$$G \times H,\quad G \oplus H,\quad \mathrm{Aut}(G),\quad \ldots$$

$$A \times B,\quad A \cup B,\quad \wp A,\quad \ldots$$

Product (binário)

Formation

product construction

$$\frac{A \text{ type} \qquad B \text{ type}}{A \times B \text{ type}} \; (\times)\text{-F}$$

Introduction

pairing

$$\frac{a : A \qquad b : B}{\langle a, b \rangle_{A \times B} : A \times B} \; (\times)\text{-I}$$

construtor de habitantes de $A \times B$

poderia ter escolhido algo menos cultural: pair a b

e botar $\langle x, y \rangle \overset{\text{sug}}{\equiv}$ pair x y

pair(a, b)

(n-ário?)

$n \geq 3$: fácil $\quad n = 0?$ $= 1?$

$(\times)$ é uma construção binária entre tipos

Como seria uma construção nulária? Assim:

$$\frac{}{\mathbb{N} : \text{Type}} \; (\mathbb{N})\text{-F}$$

Elimination

projections

$$\frac{w : A \times B}{w.l : A}\ (\times)\text{-}E_1 \qquad \frac{w : A \times B}{w.r : B}\ (\times)\text{-}E_2$$

notações alternativas:

$w \cdot 1$ $\quad w \cdot l$ $\quad \mathrm{out}l\ w$ $\quad \pi_1\ w$

$w \cdot 2$ $\quad w \cdot r$ $\quad \mathrm{out}r\ w$ $\quad \pi_2\ w$

Será que $A \times B = A$? $\quad A \times B = \mathbb{N}$?

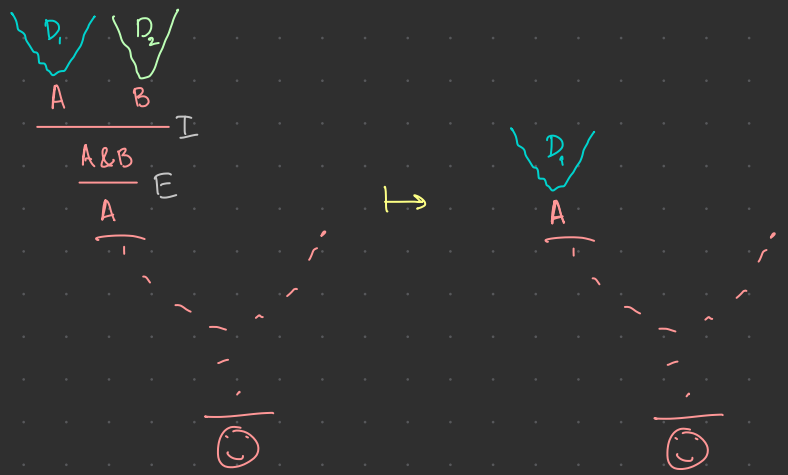
Equations

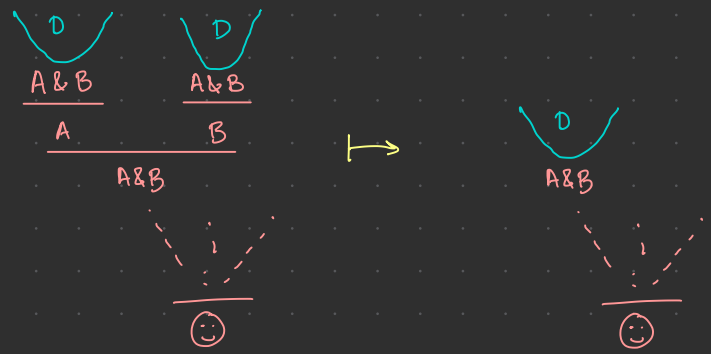
$$\langle a, b \rangle . l = a$$
$$\langle a, b \rangle . r = b$$
$$(\beta)$$

$$(w \cdot l, w \cdot r) = w \qquad (\eta)$$

$$\frac{w : A \times B \qquad B \text{ type} \qquad A \text{ type}}{w.l : A}$$

Proofs compute!





Diagramas comutativos

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"', "(\cdot l)"] \arrow[r, "\pi_2", "(\cdot r)"'] & B \\
& C \arrow[ul, bend left, "f"] \arrow[u, dashed, "p"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

o diagrama comuta

$$\Updownarrow$$

$$\pi_1 \circ p = f$$
$$\pi_2 \circ p = g$$

$$\langle \pi_1 \circ p \, , \, \pi_2 \circ p \rangle = p$$

$$\begin{tikzcd}
A \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & B \arrow[r, "f"] & C
\end{tikzcd}$$

$$f \circ i = f \circ j$$

(mas não necessariamente $i = j$)

Igualdade

lec05

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~~Definição.~~ Teorema?

Sejam $w, w' : A \times B$.

$$w =_{A\times B} w' \overset{\cancel{def}}{\cancel{\Longleftrightarrow}} \overset{?}{\Longleftrightarrow} w \cdot l =_A w' \cdot l \ \& \ w \cdot r =_B w' \cdot r$$

Demons. $(\Rightarrow)$.

$$\dfrac{\dfrac{}{w \cdot l = w \cdot l}\,\text{refl} \qquad w = w'}{w \cdot l = w' \, l}\,\text{subs}$$

similar

$w \cdot r = w' \cdot r$

$(\Leftarrow)$. Calc:

$$\begin{aligned} w &= \langle w \cdot l, w \cdot r \rangle && [(\eta)] \\ &= \langle w' \cdot l, w' \cdot r \rangle && [\text{hip} ; \text{hip}] \\ &= w' && [(\eta)] \end{aligned}$$

Categorias

Dados de uma categoria $\mathbb{C}$

$\mathbb{C}_0$ ou $\mathrm{Obj}(\mathbb{C})$: $A, B, C, \ldots$ — objetos

$\mathbb{C}_1$ ou $\mathrm{Arr}(\mathbb{C})$: $f, g, h, \ldots$ — setas

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, bend right, "j"'] & C \arrow[l, bend right, "i"'] \end{tikzcd}$$

$\mathrm{src}(f) = A$

$\mathrm{tgt}(f) = B$

$\mathrm{src}, \mathrm{tgt} : \mathrm{Arr}(\mathbb{C}) \longrightarrow \mathrm{Obj}(\mathbb{C})$

source target

Esrevemos como sinônimos:

$$\begin{cases} A \xrightarrow{f} B \\ f : A \longrightarrow B \\ f \in \mathbb{C}(A,B) \\ \mathrm{src}(f) = A \ \& \ \mathrm{tgt}(f) = B \end{cases}$$

coleção das setas de A para B

Operações

Para qualquer objeto A:

$$id_A : \mathbb{C}(A,A)$$

Para quaisquer objetos A, B, C:

$$comp_{A,B,C} : \mathbb{C}(B,C) \times \mathbb{C}(A,B) \longrightarrow \mathbb{C}(A,C)$$

Sinônimos:

$$\left\{ \begin{array}{ll} id_A & comp_{A,B,C}(f,g) \\ 1_A & f \circ g \\ & g ; f \end{array} \right.$$

Leis de categoria

(Ass) $(f \circ g) \circ h = f \circ (g \circ h)$

$$\begin{tikzcd}
A \arrow[r, "h"'] \arrow[dr, "g \circ h"'] \arrow[drr, bend right, "f \circ (g \circ h)"'] \arrow[drr, bend left, "(f \circ g) \circ h"] & B \arrow[d, "g"] \arrow[dr, "f \circ g"] & \\
& C \arrow[r, "f"] & D
\end{tikzcd}$$

(Id) $id \circ f = f$

$f \circ id = f$

$$\begin{tikzcd}
A \arrow[r, "id"] \arrow[d, bend right, "f"'] & A \arrow[d, "f"] \arrow[dr, bend left, "f"] & \\
B \arrow[r, "id"'] & B & B
\end{tikzcd}$$

Exemplos

- Tipos, funções
- Conjuntos, funções
- Proposições, demonstrações
- Grupos, homomorfismos
- Posets, funções monótonas

- 0 :
- 1 :
- 2 :
- 3 : ?

- Proposições, $(\vdash)$ $\quad a \xrightarrow{f} b$ significa $a \vdash b$
- Inteiros, $(\mid)$ $\quad a \xrightarrow{f} b$ significa $a \mid b$
- Inteiros, $(\leq)$ $\quad a \xrightarrow{f} b$ significa $a \leq b$
- Conjuntos, $(\subseteq)$ $\quad A \xrightarrow{f} B$ significa $A \subseteq B$

Universal properties : o produto

$$\forall \quad \begin{tikzcd} A & P \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\ & C \arrow[ul, "\forall f"] \arrow[u, dotted, "\exists! p"] \arrow[ur, "\forall g"'] & \end{tikzcd}$$

$$? : C \longrightarrow A \times B$$

$$\langle f, g \rangle_c$$

$\xleftarrow[\pi_1]{} P \xrightarrow[\pi_2]{}$ é um produto dos A, B
sse possui tal propriedade universal

Dualidade : coconceito

Coproduto

$$\begin{tikzcd}
A \arrow[r, "\iota_1"'] \arrow[dr, bend right, "\forall f"'] & S \arrow[d, dotted, "\exists! p"] & B \arrow[l, "\iota_2"] \arrow[dl, bend left, "\forall g"] \\
& \forall C \arrow[ul, phantom, "\shortparallel"] \arrow[ur, phantom, "\shortparallel"] &
\end{tikzcd}$$

$\xrightarrow[\iota_1]{} S \xleftarrow[\iota_2]{}$ é um coproduto dos A, B
sse possui tal propriedade universal

Coproduto: como se manifesta ...

... nos inteiros com (|):

(i) $a \mid s$ & $b \mid s$

(ii) $(\forall c)[a \mid c \ \& \ b \mid c \Rightarrow s \mid c]$

$(\forall c$ mult. com. dos $a, b)\ [\ s \mid c\ ]$

é o mmc!

... e nas outras cats?

Terminal & inicial

T terminal sse

$$\forall A \; \exists! \; A \xrightarrow{!} T$$

I coterminal sse
(inicial)

$$\forall A \; \exists! \; I \xrightarrow{!} A$$

Isomorfismos

$$A \cong B \overset{def}{\iff} (\exists f : A \to B)[\, f \text{ iso} \,]$$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f'"] \end{tikzcd}$$

$$f \text{ iso} \overset{def}{\iff} (\exists f' : A \leftarrow B)[\, f' \circ f = id_A \;\&\; f \circ f' = id_B \,]$$

"Unicidade" (a menos de isomorfismo)

$$(\forall u, v \text{ legais})[\, u \cong v \,]$$

Como entrar vs como sair

$$h : C \longrightarrow A \times B$$

$$h\ c \overset{def}{\equiv} \langle \underbrace{\overset{c}{\cdots\cdots}}_{f\,c}, \underbrace{\overset{c}{\cdots\cdots}}_{g\,c} \rangle$$

$$A \times B \longrightarrow C$$

?

(nada!)

$$f : C \to A$$

$$f\ c \overset{def}{\equiv} \underbrace{\overset{c}{\cdots\cdots}}_{f\,c}$$

$$g : C \to B$$

$$g\ c \overset{def}{\equiv} \underbrace{\overset{c}{\cdots\cdots}}_{g\,c}$$

Construções de Cats

lec06

2025-04-02

$$\mathbb{C} \longmapsto \mathbb{C}^{op}$$

Prosets & Posets vs Cats

(pre order (partial order

$(P ; \leq)$

$(\leq) : P \times P \to \mathrm{Prop}$

$$\begin{tikzcd}
 & d & \\
b & & c \\
 & a \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & d & \\
b \arrow[ur, bend left, "f"] \arrow[ur, "g"] \arrow[ur, bend right, "h"'] & & c \arrow[ul, bend right, "r"'] \\
 & a \arrow[ul, bend right, "i"'] \arrow[ul, bend left, "j"] \arrow[ur, "u"'] &
\end{tikzcd}$$

partial order

preorder

Refl : $x \leq x$

Trans : $x \leq y \ \& \ y \leq z \Rightarrow x \leq z$

Antis : $x \leq y \ \& \ y \leq x \Rightarrow x = y$

Sum (coproduct)

Formation

$$\frac{A \text{ type} \qquad B \text{ type}}{A + B \text{ type}} \quad (+)\text{-F}$$

soma

Introduction

$$\frac{a : A}{\ell_{A+B} \cdot a : A + B} \quad (+)\text{-I}_1$$

L-injection

inl a

$$\frac{b : B}{r_{A+B} \cdot b : A + B} \quad (+)\text{-I}_2$$

R-injection

inr b

Elimination

contexto

case

$$\frac{s : A + B \qquad \Gamma, a : A \vdash e_1 : C \qquad \Gamma, b : B \vdash e_2 : C}{\Gamma \vdash (\text{case } s \text{ of } \mid l \cdot a \rightsquigarrow e_1 \; ; \; r \cdot b \rightsquigarrow e_2) : C} \; (+)\text{-E}$$

$\ldots a \ldots : C \qquad \ldots b \ldots : C$

Equations

a e_1 tendo substituído todas as ocorrências livres de a por x

$$(\text{case } l \cdot x \text{ of } \mid l \cdot a \rightsquigarrow e_1 \; ; \; r \cdot b \rightsquigarrow e_2) = \boxed{e_1[a := x]}$$

$$(\text{case } r \cdot y \text{ of } \mid l \cdot a \rightsquigarrow e_1 \; ; \; r \cdot b \rightsquigarrow e_2) = e_2[b := y] \quad (\beta)$$

$$(\text{case } s \text{ of } \mid l \cdot a \rightsquigarrow l \cdot a \; ; \; r \cdot b \rightsquigarrow r \cdot b) = s \quad (\eta)$$

Currying

lec07

2025-04-02

$$\left((\alpha \times \beta) \to \gamma\right) \underset{\text{uncurry}_2}{\overset{\text{curry}_2}{\cong}} \left(\alpha \to (\beta \to \gamma)\right)$$

?

$$\text{curry}\ f \overset{\text{def}}{\equiv} \lambda a : \alpha .\ \lambda b : \beta .\ \boxed{f\ \langle a, b\rangle}$$

$$\dfrac{f : (\alpha \times \beta) \to \gamma \qquad \dfrac{a : \alpha \qquad b : \beta}{\langle a, b\rangle : \alpha \times \beta}\,(\times)\text{-I}}{f\ \langle a, b\rangle : \gamma}\,(\to)\text{-E}$$

$b : \beta$

$a : \alpha$

$f : \alpha \times \beta \to \gamma$

ou:

$$\text{curry}\ f\ a\ b \overset{\text{def}}{\equiv} f\ \langle a, b\rangle$$

curry_n , $\mathrm{uncurry}_n$: « Similar. » ? « etc. » ?

$$\mathrm{curry}_4\ f\ a\ b\ c\ d \overset{\text{def}}{\equiv} f\ \langle a,b,c,d\rangle$$

$$\mathrm{curry}_3 : A \times B \times C \longrightarrow D$$

$$\begin{tikzcd}
\mathrm{curry}_3 : (A \times B) \times C \arrow[r] \arrow[d, "\mathrm{curry}_2"] & D \\
(A \times B) \arrow[r] \arrow[d, "\mathrm{curry}_2"] & (C \to D) \\
A \to (B \to (C \to D)) &
\end{tikzcd}$$

$$\mathrm{curry}_{n+1} \overset{\text{def}}{\equiv} (\mathrm{curry}_2)^{\circ n}$$

$$\begin{tikzcd}
\mathrm{curry}_3 : A \times (B \times C) \arrow[r] \arrow[d, "\mathrm{curry}_2"] & D \\
A \to ((B \times C) \to D) \arrow[d, "?"] & \\
A \to (B \to (C \to D)) &
\end{tikzcd}$$

$$\mathrm{curry}_{n+1} \overset{\text{def}}{\equiv} ?$$

Unit

$$\frac{\text{nada}}{\text{Unit type}}\ \text{Unit-F}$$

$\mathbb{1}$ $\mathbf{1}$ 1

$$\frac{}{\star : \text{Unit}}\ \text{Unit-I}$$

☺ $()$ $\langle\rangle$

(nenhuma) Unit-E

?

β, η ?

Bool

$$\frac{}{\text{Bool type}}\ \text{Bool-F}$$

$\mathbb{B}$

$$\frac{}{ff : \text{Bool}}\ \text{Bool-I}_F \qquad \frac{}{tt : \text{Bool}}\ \text{Bool-I}_T$$

$$\frac{b : \text{Bool} \qquad y : C \qquad n : C}{\text{if } b \text{ then } y \text{ else } n : C}\ \text{Bool-E}$$

(β)

$$\text{if } tt \text{ then } y \text{ else } n = y$$

$$\text{if } ff \text{ then } y \text{ else } n = n$$

(η)

$$\text{if } b \text{ then } tt \text{ else } ff = b$$

$$
\begin{tikzcd}
\mathrm{BinProd} \arrow[r, "\text{nularizar}"] \arrow[d, "\text{dual}"'] \arrow[dr, phantom, "//"] & \mathrm{Unit} \arrow[d, "\text{dual}"] \\
\mathrm{BinSum} \arrow[r, "\text{nularizar}"] & \boxed{\begin{array}{c}\mathrm{Counit} \\ \parallel \\ \mathrm{Empty}\end{array}}
\end{tikzcd}
$$

Empty

$$\frac{}{\mathrm{Empty}\ \mathrm{type}}\ \mathrm{F}$$

$0\ 1\ 0$

Nenhuma - I

$$\frac{e : \mathrm{Empty}}{\mathrm{boom}_C\ e \quad : C}\ \mathrm{E}$$

abort_C

Quem é o void da C?

β, η ?

Funções de graça

Tendo... será que ganho ..

$$C \to C'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B) \to (C' \to D)) \to (E + (D \to G)))
\end{tikzcd}$$

$$B \to B'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B') \to (C \to D)) \to (E + (D \to G)))
\end{tikzcd}$$

$$D \to D'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B) \to (C \to D')) \to (E + (D' \to G)))
\end{tikzcd}$$

Quem é o void da C?

lec08

2025-04-09

```
int f(void)
    return 42;
```

```
void g(int x)
    return;
```

$$\frac{g : \mathrm{int} \to \mathrm{void} \qquad 42 : \mathrm{int}}{g(42) : \mathrm{void}}$$

$f(a_1, \ldots, a_n)$

$f\,\langle a_1, \ldots, a_n \rangle$

$f\,a_1\,\ldots\,a_n$

n-ária

n-ária descurrificada

n-ária currificada

$f()$ ← dá falsa impressão que f recebeu nada

$f\,()$ ← deixa claro que f recebeu ()

o único habitante do $\mathbb{1}$

Function space

$$\frac{A \text{ type} \qquad B \text{ type}}{A \to B \text{ type}} \; (\to)\text{-F}$$

$$\frac{\Gamma, a : A \vdash b : B}{\Gamma \vdash \lambda a : A . b : A \to B} \; (\to)\text{-I}$$

$$\frac{f : A \to B \qquad a : A}{f \sqcup a : B} \; (\to)\text{-E}$$

β? η?

Exemplo:

$$\frac{w : \mathrm{Int},\ u : \mathrm{Int},\ a : \mathrm{Int} \vdash a + 2 \cdot u : \mathrm{Int}}{w : \mathrm{Int},\ u : \mathrm{Int} \vdash \lambda a : \alpha .\ a + 2 \cdot u : \mathrm{Int} \to \mathrm{Int}}$$

Regras de inferência

$$\frac{J_1 \quad \cdots \quad J_n}{J}\ R$$

premissas (0 ou mais)

nome da regra (rótulo)

juízos (judgements)

conclusão (exatamente 1)

Contexts

implícito: $a : A \vdash B \text{ type}$

$$a : A, \quad b : B, \quad c : C, \quad \ldots$$

implícito: $\vdash A \text{ type}$

implícito: $a : A, b : B \vdash C \text{ type}$

$$a_1 : A_1, \quad a_2 : A_2(a_1), \quad a_3 : A_3(a_1, a_2), \quad \ldots$$

Escrevendo: $n : \mathbb{N}, \quad u : \mathbb{R}^n, \quad v : \mathbb{R}^n, \quad m : \mathbb{N}, \quad A : \mathbb{Z}^{n \times m}$

Implícitos: $\vdash \mathbb{N} \text{ type}$

$n : \mathbb{N} \vdash \mathbb{R}^n \text{ type}$

$n : \mathbb{N}, u : \mathbb{R}^n \vdash \mathbb{R}^n \text{ type}$

$n : \mathbb{N}, u : \mathbb{R}^n, v : \mathbb{R}^n \vdash \mathbb{N} \text{ type}$

$n : \mathbb{N}, u : \mathbb{R}^n, v : \mathbb{R}^n, m : \mathbb{N} \vdash \mathbb{Z}^{n \times m} \text{ type}$

Dependent types / Families

"$A : I \to \mathrm{Type}$"

indexing

$i : I \vdash B(i)$ type

$i : I \vdash B(i)$ type

B é uma família de tipos (indexada por $i : I$)

B é um tipo dependente (do $i : I$)

Exemplo:

"$\mathbb{R}^{\square} : \mathbb{N} \to \mathrm{Type}$"

$n : \mathbb{N} \vdash \mathbb{R}^n$ type

Exemplo:

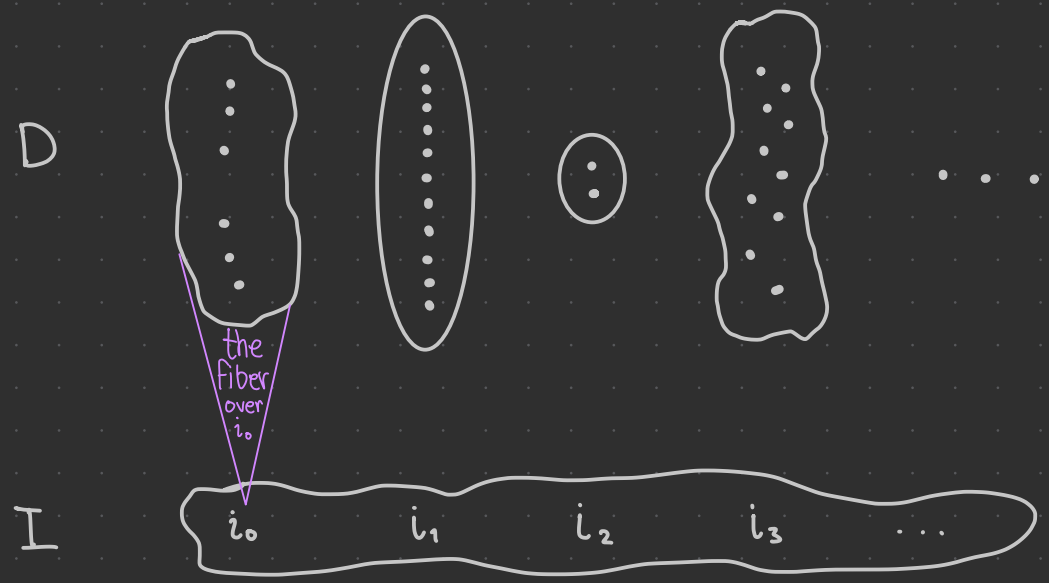
"$A : \mathbb{B} \to \mathrm{Type}$"

$A(\mathbb{f}) \overset{\text{def}}{\equiv} \mathbb{N}$

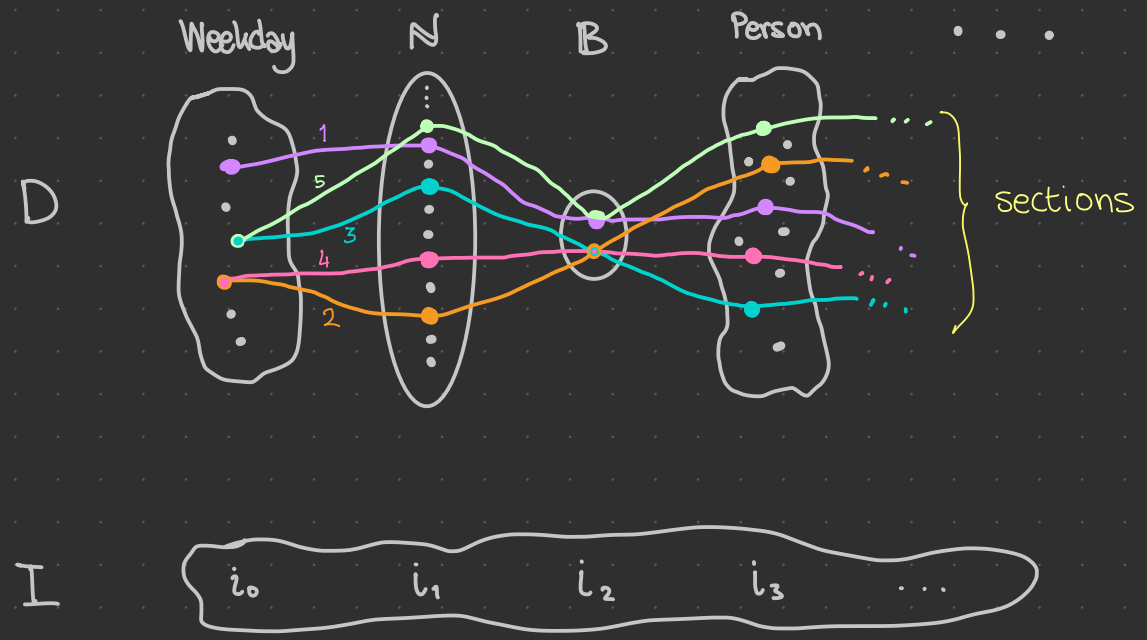
$A(\mathbb{t}) \overset{\text{def}}{\equiv} \mathbb{N} \times (\mathbb{N} \to \mathbb{N})$

$$\boxed{b : \mathbb{B} \vdash A(b) \text{ type}}$$

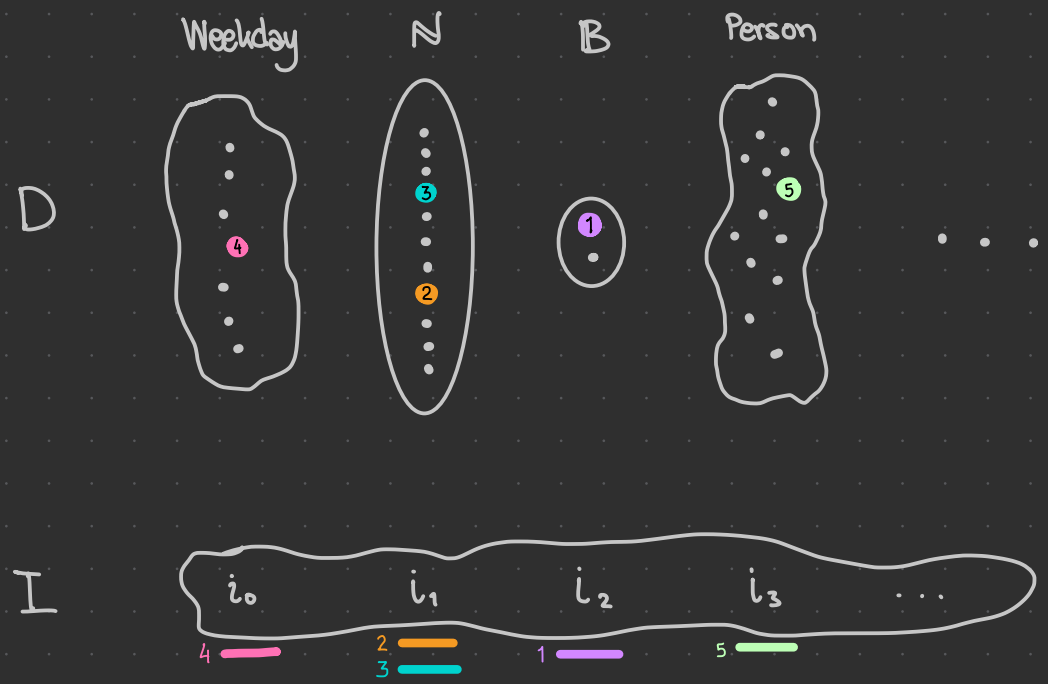
$$i : I \vdash D(i) \text{ type}$$



5 habitantes do TD



5 habitantes do ΣD



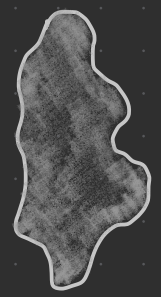
$$b : \mathbb{B} \vdash P(b) \text{ type}$$

	$P(ff)$	$P(tt)$
	$\mathbb{N}$	$\mathbb{N} \to \mathbb{N}$
P	⋮ 5 4 3 2 1 0	⋮ fib fact (4 ·) (+ 3) double succ id
$\mathbb{B}$	ff	tt

$$n : \mathbb{N} \vdash U(n) \text{ type}$$

	$U(0) = \{0\}$	$U(1) = [0,1]$	$U(2) = [0,1]^2$	...
U	⊙	⊙	⊡	
$\mathbb{N}$	0	1	2	...

$$b : \mathbb{B} \vdash A(b) \text{ type}$$

$A(b:\mathbb{B})$	⋮ 5 4 3 2 1 0	
$\mathbb{B}$	ff	tt

Π (Dependent product)

A é um dependent type

$$\frac{\boxed{i : I \vdash A(i) \ \text{type}}}{\vdash \prod_{i:I} A(i) \ \text{type}} \ (\Pi)\text{-F}$$

$\Pi(i:I).A(i)$

$(i:I) \to A(i)$

$? : (b:\mathbb{B}) \to A(b)$

I? E? β, η?

Σ (Dependent sum)

$$\frac{i : I \vdash A(i) \ \text{type}}{\vdash \sum_{i:I} A(i) \ \text{type}} \ (\Sigma)\text{-F}$$

$$\Sigma(i:I).A(i)$$

$$(i : I) \times A(i)$$

$$? : (b : \mathbb{B}) \times A(b)$$

I? E? β, η?

Isomorfismos

lec09

2025-04-14

$$\alpha \cong \beta \overset{def}{\iff} (\exists f : \alpha \to \beta)[\, f \text{ iso} \,]$$

invertível

$$f : \alpha \to \beta \text{ iso} \overset{def}{\iff} (\exists f' : \alpha \leftarrow \beta)[\, \overbrace{f' \circ f = id_\alpha \;\&\; f \circ f' = id_\beta}^{f' \text{ inversa de } f} \,]$$

$$\alpha \xrightarrow{\cong} \beta$$

$$f : \alpha \cong \beta$$

$$f^{-1} \overset{def}{\equiv} \boxed{\text{a}} \text{ inversa de } f$$

Θ. $(\cong)$ é uma relação de equivalência:

$(\cong)$ - refl

$(\cong)$ - trans

$(\cong)$ - sym

Aritmética de tipos

$$(\alpha + \beta) + \gamma \cong \alpha + (\beta + \gamma)$$

$$\mathbb{0} + \alpha \cong \alpha$$

$$\alpha + \beta \cong \beta + \alpha$$

$$(\alpha \times \beta) \times \gamma \cong \alpha \times (\beta \times \gamma)$$

$$\mathbb{1} \times \alpha \cong \alpha$$

$$\alpha \times \beta \cong \beta \times \alpha$$

Distributividades:

$$\delta \times (\alpha + \beta) \cong (\delta \times \alpha) + (\delta \times \beta)$$

$$\delta \times \mathbb{0} \cong \mathbb{0}$$

$$\delta \times \sum_{i:I} \alpha_i \cong \sum_{i:I} (\delta \times \alpha_i)$$

Exemplo:

$$f : \delta \times (\alpha + \beta) \cong (\delta \times \alpha) + (\delta \times \beta)$$

$$f\ \langle d, \ell \cdot a \rangle \overset{\text{def}}{\equiv} \ell \cdot \langle d, a \rangle$$

$$f\ \langle d, r \cdot b \rangle \overset{\text{def}}{\equiv} r \cdot \langle d, b \rangle$$

Sem pattern-matching:

$$f\ w \overset{\text{def}}{\equiv} \text{case } w.r \text{ of } \left[\begin{array}{l} \ell\ a \rightsquigarrow \ell \cdot \langle w \cdot \ell, a \rangle \\ r \cdot b \rightsquigarrow r \cdot \langle w \cdot \ell, b \rangle \end{array} \right]$$

Conclusão: ❤ pattern-matching

Exponenciação

$$\alpha^{\beta+\gamma} \cong \alpha^{\beta} \times \alpha^{\gamma}$$

versão I-ária

$$\alpha^{\sum_{i:I} \beta_i} \cong \prod_{i:I} \alpha^{\beta_i}$$

versão nulária

$$\alpha^{\mathbb{0}} \cong \mathbb{1}$$

$$\alpha^{\beta\times\gamma} \underset{\text{currificação}}{\cong} (\alpha^{\beta})^{\gamma}$$

Conclusão: Aprendi currificação no ensino médio !?

$$\alpha \; ? \; (\beta+\gamma) \cong (\alpha \; ? \; \beta) \times (\alpha \; ? \; \gamma)$$

$$\alpha \; ? \; (\beta\times\gamma) \cong (\alpha \; ? \; \beta) \; ? \; \gamma$$

$$\boxed{(?) \equiv (\leftarrow)}$$

$$\alpha^{\beta} \overset{\text{def}}{\equiv} \beta \to \alpha \quad \text{ou} \quad \alpha \leftarrow \beta \quad \text{ou} \quad \alpha^{\swarrow \beta}$$

Cardinalidades

$$|\alpha \times \beta| = |\alpha| \cdot |\beta|$$

$$|\alpha + \beta| = |\alpha| + |\beta|$$

$$|\alpha \leftarrow \beta| = |\alpha|^{|\beta|}$$

$$2 \overset{?}{\underset{\text{def}}{\equiv}} 1 + 1 \cong \mathbb{B}$$

$$3 \cong 2 + 1$$

$$\vdots$$

$$n$$

$$\alpha^0 \cong 1$$

$$\alpha^1 \cong \alpha$$

$$\alpha^2 \cong \alpha \times \alpha$$

$$\vdots$$

$$(\alpha + \beta)^2 \cong ?$$

$A + B$ — Caso binário: $I :\equiv 2$, $D(0) :\equiv A$; $D(1) :\equiv B$

Σ — família invariável: $(I :\equiv A;\ D(a) :\equiv B)$

$A \times B$ — Caso binário: $I :\equiv 2$, $D(0) :\equiv A$; $D(1) :\equiv B$

Π — família invariável: $(I :\equiv A;\ D(a) :\equiv B)$

$A \to B$

família invariável: $I :\equiv A$; $D(a) :\equiv B$

$$\sum_{(a:A)} D(a) \equiv (a:A) \times B \equiv \boxed{A \times B} \qquad \prod_{(a:A)} D(a) \equiv (a:A) \to B \equiv \boxed{A \to B}$$

caso binário: $I :\equiv 2$; $D(0) :\equiv A$; $D(1) :\equiv B$

$$\sum_{(i:2)} D(i) \cong D(0) + D(1) \equiv \boxed{A + B} \qquad \prod_{(i:2)} D(i) \cong D(0) \times D(1) \equiv \boxed{A \times B}$$

Propositions as Types (Curry-Howard)

Certos tipos podem ser vistos como proposições e seus habitantes (as construções dos) como evidências - demonstrações.

Logic	Types
$\bot$	$\mathbb{0}$
$\top$	$\mathbb{1}$
$P \wedge Q$	$P \times Q$
$P \Rightarrow Q$	$P \to Q$
$P \vee Q$	$P + Q$
$\forall x \in A.\ P(x)$	$\prod(x:A).\ P(x)$
$\exists x \in A.\ P(x)$	$\sum(x:A).\ P(x)$
$u \overset{?}{=}_A v$	$Id_A(u,v)$

decisão

disjunção

construção

existencial

as identificações dos $u, v : A$

$\neg P \equiv P \Rightarrow \bot$

$P \Leftrightarrow Q \equiv \ldots$

truncamento

$\| P \| \overset{def}{\equiv} \neg\neg P$

Previously ... Funções de graça

lec10

2025-04-16

Tendo... será que ganho ..

$$C \downarrow C'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B) \to (C' \to D)) \to (E + (D \to G)))
\end{tikzcd}$$

$$B \downarrow B'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B') \to (C \to D)) \to (E + (D \to G)))
\end{tikzcd}$$

$$D \downarrow D'$$

$$\begin{tikzcd}
(((A \times B) \to (C \to D)) \to (E + (D \to G))) \arrow[d, bend right, "?"'] \\
(((A \times B) \to (C \to D')) \to (E + (D' \to G)))
\end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times B \arrow[d, "?"] \\ A' & A' \times B \end{tikzcd} \qquad (a, b) \mapsto (f\,a, b)$$

$$\begin{tikzcd} B \arrow[d, "g"] & A \times B \arrow[d, "?"] \\ B' & A \times B' \end{tikzcd}$$

Similar

$$\begin{tikzcd} A \arrow[d, "f"'] & A + B \arrow[d, "?"] \\ A' & A' + B \end{tikzcd} \qquad \begin{aligned} (l \cdot a) &\mapsto l \cdot f\,a \\ (r \cdot b) &\mapsto r \cdot b \end{aligned}$$

$$\begin{tikzcd} B \arrow[d, "g"] & A \times B \arrow[d, "?"] \\ B' & A \times B' \end{tikzcd}$$

Similar

$$\begin{tikzcd} A \arrow[d, "f"'] & A \to B \arrow[d, "?"] \arrow[d, "?"', "u'"] \\ A' & A' \to B \end{tikzcd} \qquad \begin{aligned} &\text{pre comp.} \\ &(\circ f) \\ u' &\mapsto \lambda a.\, u'(f\,a) \end{aligned}$$

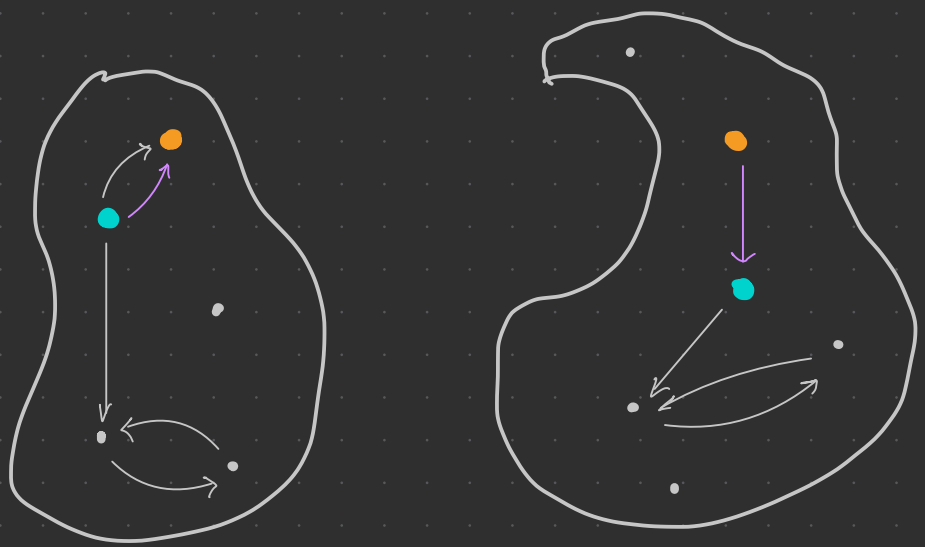
$$\begin{tikzcd} B \arrow[d, "g"'] & A \to B \arrow[d, "?"] \\ B' & A \to B' \end{tikzcd} \qquad \begin{aligned} &\text{post-comp} \\ &(g \circ) \\ u &\mapsto \lambda a.\, \underbrace{g\,(u\,a)}_{(g \circ u)\,a} \end{aligned}$$

Cats

$$\mathbb{C} \xrightarrow{F} \mathbb{D}$$

functor (covariant)

(contravariant)



$$\mathrm{src} \circ F = F \circ \mathrm{src}$$

$$\mathrm{tgt} \circ F = F \circ \mathrm{tgt}$$

$$F(g \circ f) = Fg \circ Ff$$

$$F(\mathrm{id}_A) = \mathrm{id}_{FA}$$

$$F_0 : \mathbb{C}_0 \longrightarrow \mathbb{D}_0$$

$$F_1 : \mathbb{C}_1 \longrightarrow \mathbb{D}_1$$

comp. respeita functorialidade

hw

class Functor F : Ty → Ty

fmap : $(A \xrightarrow{f} B) \xrightarrow{F} (F\,A \xrightarrow{F\,f} F\,B)$

(+ leis)

$$\text{Int} \longmapsto \text{List Int}$$

$$(\text{Int} \to \text{List Int}) \xmapsto{\text{List}} \text{List}\,(\text{Int} \to \text{List Int})$$

↳ Functor List sendo aplicado em objetos

instance Functor List : Ty → Ty

fmap : (A → B) → (List A → List B)

List

List_0 : Type → Type

List_1 : $(A \to B) \to (L_0 A \to L_0 B)$

fmap

$$D \xmapsto{(A \times -)} A \times D$$

$$\begin{tikzcd} D \arrow[d, "f"'] \\ D' \end{tikzcd} \xmapsto{(A \times -)} \begin{tikzcd} A \times D \arrow[d, "A \times f"] \\ A \times D' \end{tikzcd}$$

$(\times)$: covariante

$(+)$: covariante

$(_ \to B)$: contravariante

$(A \to _)$: covariante

Σ, Π

$$\frac{\Gamma \vdash I \ \mathsf{type} \qquad \Gamma, i : I \vdash A_i \ \mathsf{type}}{\Gamma \vdash (i : I) \times A_i \ \mathsf{type}} \ \Sigma\text{-F}$$

ou $\sum_{i:I} A_i$ ou $\Sigma\, i : I .\, A_i$

$$\frac{\Gamma \vdash I \ \mathsf{type} \qquad \Gamma, i : I \vdash A_i \ \mathsf{type}}{\Gamma \vdash (i : I) \to A_i \ \mathsf{type}} \ \Pi\text{-F}$$

ou $\prod_{i:I} A_i$ ou $\Pi\, i : I .\, A_i$

$$\frac{\Gamma \vdash i : I \qquad \Gamma \vdash a_i : A_i}{\Gamma \vdash \langle i, a_i \rangle : (i : I) \times A_i} \; \Sigma\text{-I}$$

$$\frac{\Gamma, i : I \vdash a_i : A_i}{\Gamma \vdash \lambda i : I.\, a_i : (i : I) \to A_i} \; \Pi\text{-I}$$

$$\frac{\Gamma \vdash w : (i : I) \times A_i}{\begin{array}{c}\Gamma \vdash w.l : I \\ \Gamma \vdash w.r : A_{w.l}\end{array}} \; \Sigma\text{-E}$$

$$\frac{\begin{array}{c}\Gamma \vdash f : (i : I) \to A_i \\ \Gamma \vdash j : I\end{array}}{\Gamma \vdash f\,_\,j : A_j} \; \Pi\text{-E}$$

Universos

Universe : um tipo cujos elementos são tipos.

$\cancel{\mathrm{Type} : \mathrm{Type}}$ Girard

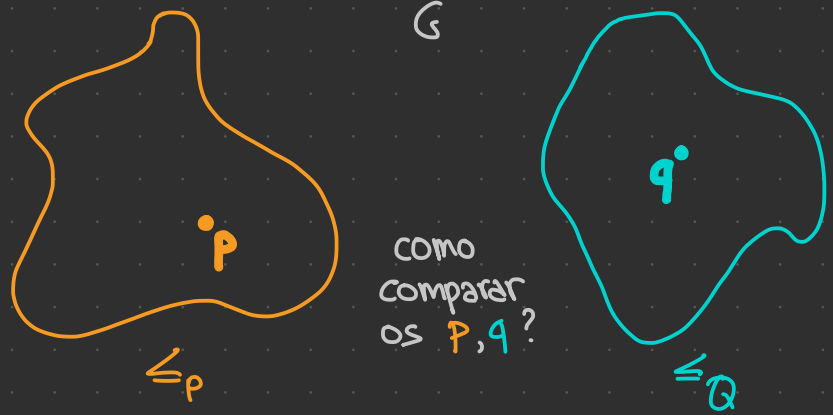
quão longa?

$\mathrm{Type}_0 : \mathrm{Type}_1 : \mathrm{Type}_2 : \ldots$

Galois connections

lec11
2025-04-23

$$P \underset{G}{\overset{F}{\rightleftarrows}} Q$$



Quais são as setas?

Funções monótonas:

$$x \leq_P y \Rightarrow f\,x \leq_Q f\,y$$

Galois (connection)

$$F \;\text{☺}\; G \overset{\text{def}}{\Longleftrightarrow} F\,p \leq_Q q \Leftrightarrow p \leq_P G\,q$$

lower ou left — adjoint — upper ou right

Θ. "dado um dos dois, o outro é determinado"

$$\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "\text{ceil?}"] \arrow[r, bend right, "\text{floor?}"'] & \mathbb{Z} \arrow[l, bend left, "\text{inc}"] \end{tikzcd}$$

inc ☺ ?

? ☺ inc

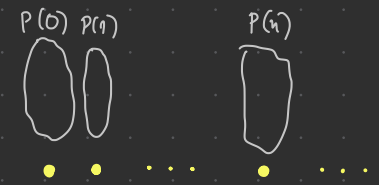
$\mathbb{N}$

$$\frac{}{\mathbb{N}\ \text{type}}\ \mathbb{N}\text{-F}$$

$$\frac{}{0 : \mathbb{N}}\ \mathbb{N}\text{-I}_z$$

$$\frac{n : \mathbb{N}}{\text{succ}(n) : \mathbb{N}}\ \mathbb{N}\text{-I}_s$$

$$\frac{\begin{array}{l}\Gamma \vdash P : \mathbb{N} \to \text{Type} \\ \Gamma \vdash b : P(0) \\ \Gamma, k : \mathbb{N}, h : P(k) \vdash i(k,h) : P(\text{succ}(k)) \\ \Gamma \vdash n : \mathbb{N}\end{array}}{\Gamma \vdash \text{ind}_{\mathbb{N}}(P, b, i, n) : P(n)}\ \mathbb{N}\text{-E}$$



Equações:

$$\text{ind}_{\mathbb{N}}(P, b, i, 0) = b \quad : P(0)$$

$$\text{ind}_{\mathbb{N}}(P, b, i, \text{succ}(k)) = i(k, \text{ind}_{\mathbb{N}}(P, b, i, k)) : P(\text{succ}(k))$$

MLTT

Martin Löf Type Theory

lec12

2025-04-28

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{C}\ \text{label}$$

ítélet

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash f : A \to B}{\Gamma \vdash f\,a : B}$$

(i) A é um tipo bem formada dentro do contexto Γ.

$$\Gamma \vdash A \text{ type}$$

judgementally equal

(ii) A, B são tipos juizamente iguais dentro do contexto Γ.

$$\Gamma \vdash A \doteq B \text{ type}$$

(iii) a é um elemento do tipo A no ctx Γ

$$\Gamma \vdash a : A$$

(iv) a, b são elementos jmente iguais do tipo A no Γ

$$\Gamma \vdash a \doteq b : A$$

$$\Gamma \vdash J$$

Contexto:

$x_1 : A_1, \quad x_2 : A_2(x_1), \quad x_3 : A_3(x_1, x_2), \quad \ldots, \quad x_n : A_n(x_1, \ldots, x_{n-1})$

Para cada $1 \leq k \leq n$:

$$x_1 : A_1, \ldots, x_{k-1} : A_{k-1}(x_1, \ldots, x_{k-2}) \vdash A_k(x_1, \ldots, x_{k-1}) \text{ type}$$

Nomes únicos!

Processo recursivo de verificação de ctx (de L para R).

Famílias de tipos

$\Gamma, x : A \vdash B(x)$ type

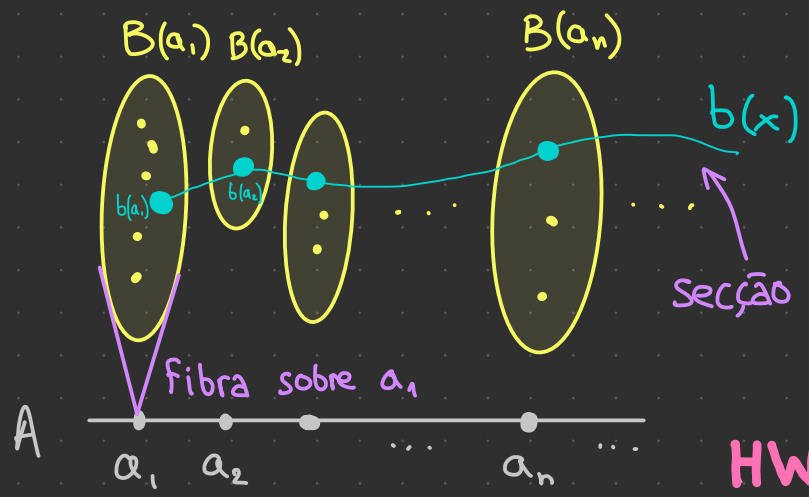
$A \to \text{Type}$

B é uma fam de tipos indexada por $x : A$

Exemplo:

$n : \mathbb{N} \vdash \mathbb{R}^n$ type

$\Gamma, x : A \vdash b(x) : B(x)$



HW

Regras

(i) Sobre info implícitas de formação de: ctx, tipos, elems

$$\frac{\Gamma, x : A \vdash B(x)\ \mathrm{type}}{\Gamma \vdash A\ \mathrm{type}}\ \text{idx-type}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash A\ \mathrm{type}}\ \text{elem-type}$$

$$\frac{\Gamma \vdash A \doteq B\ \mathrm{type}}{\begin{array}{l}\Gamma \vdash A\ \mathrm{type} \\ \Gamma \vdash B\ \mathrm{type}\end{array}}\ \begin{array}{l}\text{jeq-L} \\ \text{typejeq-type} \\ \text{-L} \\ \text{-R}\end{array}$$

$$\frac{\Gamma \vdash a \doteq b : A}{\begin{array}{l}\Gamma \vdash a : A \\ \Gamma \vdash b : A\end{array}}\ \begin{array}{l}\text{jeq-L} \\ \text{elemjeq-elem} \\ \text{-L} \\ \text{-R}\end{array}$$

HW

$$\frac{\Gamma, x : A, \Delta \vdash J}{\Gamma \vdash A\ \mathrm{type}}\ \begin{array}{l}\text{ctx} \\ \text{-type}\end{array}$$

$$\frac{\dfrac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash a : A}}{\Gamma \vdash A\ \mathrm{type}}$$

derivável:

$$\frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash A\ \mathrm{type}}\ \text{elemjeq-type}$$

(ii) Que estabelecem que $(\doteq)$ é uma rel de equivalência

typejeq-		elemjeq-
$\dfrac{\Gamma \vdash A \ \mathsf{type}}{\Gamma \vdash A \doteq A \ \mathsf{type}}\ \text{jeq-refl}$	-refl	$\dfrac{\Gamma \vdash a : A}{\Gamma \vdash a \doteq a : A}\ \text{jeq-refl}$
$\dfrac{\Gamma \vdash A \doteq B \ \mathsf{type} \qquad \Gamma \vdash B \doteq C \ \mathsf{type}}{\Gamma \vdash A \doteq C \ \mathsf{type}}\ \text{jeq-trans}$	-trans	$\dfrac{\Gamma \vdash a \doteq b : A \qquad \Gamma \vdash b \doteq c : A}{\Gamma \vdash a \doteq c : A}\ \text{jeq-trans}$
$\dfrac{\Gamma \vdash A \doteq B \ \mathsf{type}}{\Gamma \vdash B \doteq A \ \mathsf{type}}\ \text{jeq-sym}$	-sym	$\dfrac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash b \doteq a : A}\ \text{jeq-sym}$

(iii) Conversões

scheme: (4-in-1)

conv-var conversão de variável

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, x : A', \Delta \vdash J}\ \text{conv}(x : A \rightsquigarrow A')\text{-var}$$

Deriváveis: **(HW)**

conv-elem conversão de elemento

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \qquad \Gamma \vdash a : A}{\Gamma \vdash a : A'}\ \text{conv}(a : A \rightsquigarrow A')\text{-elem}$$

conv-elemjeq conversão de elemjeq

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \qquad \Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A'}\ \text{conv}(a \doteq b : A \rightsquigarrow A')\text{-elemjeq}$$

(iv) Substituições

Também usado, mas vamos evitar: $e[a/x]$

$$e[x := a] \stackrel{\text{def}}{\equiv} \begin{cases} e \text{ substituindo todas as} \\ \text{ocorrências de } x \text{ por } a \\ \text{tomando os devidos cuidados} \end{cases}$$

sub-elem

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, \Delta[x := a] \vdash J[x := a]} \; \text{sub}(x:=a)\text{-elem}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, y_1 : B_1, \ldots, y_n : B_n \vdash C \text{ type}}{\Gamma, y_1 : B_1[x := a], \ldots, y_n : B_n[x := a] \vdash C[x := a] \text{ type}}$$

sub-typeeq

$$\frac{\Gamma \vdash a \doteq a' : A \qquad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[x := a] \vdash B[x := a] \doteq B[x := a'] \text{ type}} \; \text{sub}(x:=a, a')\text{-typeeq}$$

sub-elemeq

$$\frac{\Gamma \vdash a \doteq a' : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[x := a] \vdash b[x := a] \doteq b[x := a'] : B[x := a]} \; \text{sub}(x:=a, a')\text{-elemeq}$$

sub-typejeq & sub-elemjeq as congruence rules

cong-jeq-type (sub-typejeq)

$$\frac{a \doteq a' : A \qquad B(x:A)\ \mathsf{type}}{B(a) \doteq B(a')\ \mathsf{type}}\ \text{cong-jeq}(B[x := a, a'])\text{-type}$$

cong-jeq-elem (sub-elemjeq)

$$\frac{a \doteq a' : A \qquad b(x:A) : B(x:A)}{b(a) \doteq b(a') : B(a)}\ \text{cong-jeq}(b[x := a, a'])\text{-elem}$$

(v) Weakening

lec13
2025-04-30

weak

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, \Delta \vdash J}{\Gamma, x : A, \Delta \vdash J}\ \text{weak}(\Gamma; x : A) * \ \text{wkg}$$

x fresco (em Γ, Δ)

CASO ESPECIAL:

kfam

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma, x : A \vdash B \text{ type}}\ \text{kfam}(x : A, B) *$$

(formação de família constante (trivialmente dependente)

...finalmente:

(vi) generalized elem, identity, ⊢-refl, "axiom", ...

gen

$$\frac{\Gamma \vdash A \ \mathrm{type}}{\Gamma, x : A \vdash x : A}\ \begin{array}{l}\mathrm{gen}(x:A)\\ \delta(x:A)\end{array} \quad *$$

Derivações

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{C}$$

rename

$$\frac{\Gamma,\ x : A,\ \Delta \vdash J}{\Gamma,\ x' : A,\ \Delta[x := x'] \vdash J[x := x']}\ \text{rename}(x, x')\ *$$

x' fresco (no ctx)

HW

$$\frac{\overset{?}{\Gamma \vdash A \ \text{type}}}{\Gamma,\ x' : A \vdash x' : A}\ \text{gen}(x' : A) \qquad \frac{\overset{?}{\Gamma \vdash A \ \text{type}} \qquad \Gamma,\ x : A,\ \Delta \vdash J}{\Gamma,\ x' : A,\ x : A,\ \Delta \vdash J}\ \overset{*}{w}(\Gamma; x' : A)$$

$$\frac{\Gamma,\ x' : A \vdash x' : A \qquad \Gamma,\ x' : A,\ x : A,\ \Delta \vdash J}{\Gamma,\ x' : A,\ \Delta[x := x'] \vdash J[x := x']}\ \text{sub}_{\text{-elem}}(x := x')$$

swap

Qual é o papel disso?

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x:A, y:B, \Delta \vdash J}{\Gamma, y:B, x:A, \Delta \vdash J}\ \mathsf{swap}(x,y)$$

$$\frac{\dfrac{\Gamma \vdash B \text{ type}}{\Gamma, y:B \vdash y:B}\ \delta(y:B) \qquad \dfrac{\Gamma \vdash B \text{ type} \qquad \dfrac{\Gamma, x:A, y:B, \Delta \vdash J}{\Gamma, x:A, y':B, \Delta[y:=y'] \vdash J[y:=y']}\ \mathsf{rename}(y,y')\ *}{\Gamma, y:B, x:A, y':B, \Delta[y:=y'] \vdash J[y:=y']}\ \mathsf{weak}(\Gamma; y:B)}{\Gamma, y:B, x:A, \boxed{\Delta} \vdash \boxed{J}}\ \mathsf{sub}_{\text{-elem}}(y':=y)$$

$$\boxed{\Delta[y:=y'][y':=y]} \quad * \qquad \boxed{J[y:=y'][y':=y]} \quad *$$

versões $\leftarrow$ e -R

conv-var$\leftarrow$

$$\frac{\Gamma \vdash A' \doteq A \ \mathsf{type} \qquad \Gamma, x : A, \Delta \vdash J}{\Gamma, x : A', \Delta \vdash J}$$

conv $(x : A' \leftarrow A)$
-var
conv $(x : A \rightarrow A')$

conv-elem$\leftarrow$

conv $(a : A' \leftarrow A)$
conv $(a : A \rightarrow A')$

As: typejeq-type-R, elemjeq-elem-R são deriváveis.

conv-elemjeq$\leftarrow$

conv $(a \doteq b : A' \leftarrow A)$
conv $(a \doteq b : A \rightarrow A')$

HW

conv-elem conversão de elemento

$$\frac{\Gamma \vdash A \doteq A' \ \text{type} \qquad \Gamma \vdash a : A}{\Gamma \vdash a : A'} \ \text{conv-elem}\,(a : A \rightsquigarrow A')$$

conv-elemjeq conversão de elemjeq

$$\frac{\Gamma \vdash A \doteq A' \ \text{type} \qquad \Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A'} \ \text{conv-elemjeq}\,(a \doteq b : A \rightsquigarrow A')$$

ctx-last

$$\frac{\Gamma, x : A \vdash J}{\Gamma \vdash A \ \text{type}} \ \text{ctx-last}$$

ctx type-validity of context

$$\frac{\Gamma, x : A, \Delta \vdash J}{\Gamma \vdash A \ \text{type}} \ \text{ctx}$$

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conv-elem

$$\frac{\Gamma \vdash A \doteq A' \ \mathsf{type} \qquad \Gamma \vdash a : A}{\Gamma \vdash a : A'} \ \mathsf{conv}\text{-}\mathsf{elem}\ (a : A \rightsquigarrow A')$$

$$\frac{\Gamma \vdash A \doteq A' \ \mathsf{type}}{\Gamma \vdash A' \ \mathsf{type}} \ \mathsf{jeq}\text{-}\mathsf{R}$$

$$\frac{\Gamma \vdash A' \ \mathsf{type}}{\Gamma, x : A' \vdash x : A'} \ \mathsf{gen}\ (x : A')$$

$$\frac{\Gamma \vdash A \doteq A' \ \mathsf{type} \qquad \Gamma, x : A' \vdash x : A'}{\Gamma, x . A \vdash x : A'} \ \mathsf{conv}^{\leftarrow}\text{-}\mathsf{var}\ (x : A' \rightsquigarrow A)$$

$$\frac{\Gamma \vdash a . A \qquad \Gamma, x . A \vdash x : A'}{\Gamma \vdash \boxed{a : A'}} \ \mathsf{sub}\text{-}\mathsf{elem}\ (x := a)$$

$$\boxed{(x \cdot A')[x := a]}$$

conv-elemjeq

$$\frac{\Gamma \vdash A \doteq A' \ \mathsf{type} \qquad \Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A'} \ \mathsf{conv}\text{-}\mathsf{elemjeq}\,(a \doteq b : A \rightsquigarrow A')$$

$$\frac{\Gamma \vdash A \doteq A' \ \mathsf{type} \qquad \underbrace{\dfrac{\dfrac{\Gamma \vdash A \doteq A' \ \mathsf{type}}{\Gamma \vdash A' \ \mathsf{type}}\ \mathsf{jeq\text{-}R}}{\Gamma, x : A' \vdash x : A'}\ \mathsf{gen}\,(x : A')}_{\Gamma, x : A \vdash x : A'}}{\Gamma \vdash a \doteq b : A} \ \mathsf{conv}^{\leftarrow}_{\mathsf{var}}\,(x : A' \rightsquigarrow A) \ \ \mathsf{sub\text{-}ejeq}\,(x := a, b)$$

$$\Gamma \vdash \boxed{a} \doteq \boxed{b} : \boxed{A'}$$

$\boxed{x[x := a]}$ $\boxed{x[x := b]}$ $\boxed{A'[x := a]}$

ctx-last

$$\frac{\Gamma, x : A \vdash J}{\Gamma \vdash A \ \mathrm{type}}\ \text{ctx-last}$$

$$\frac{\Gamma, x : A \vdash B \ \mathrm{type}}{\Gamma \vdash A \ \mathrm{type}}\ \text{idx-type}$$

$$\frac{\dfrac{\Gamma, x : A \vdash B \doteq B' \ \mathrm{type}}{\Gamma, x : A \vdash B \ \mathrm{type}}\ \text{jeq-L}}{\Gamma \vdash A \ \mathrm{type}}\ \text{idx-type}$$

$$\frac{\dfrac{\Gamma, x : A \vdash b : B}{\Gamma, x : A \vdash B \ \mathrm{type}}\ \text{elem-type}}{\Gamma \vdash A \ \mathrm{type}}\ \text{idx-type}$$

$$\frac{\dfrac{\Gamma, x : A \vdash b \doteq b' : B}{\Gamma, x : A \vdash B \ \mathrm{type}}\ \text{ejeq-type}}{\Gamma \vdash A \ \mathrm{type}}\ \text{idx-type}$$

ctx

$$\frac{\Gamma, x : A, \Delta \vdash J}{\Gamma \vdash A \text{ type}}\ \text{ctx}$$

Por indução (recursão?) no Δ:

BASE ($\Delta =: [\,]$):

(ctx-last)

PASSO INDUTIVO ($\Delta =: \Delta', z : Z$):

$$\cfrac{\cfrac{\Gamma, x : A, \Delta', z : Z \vdash J}{\Gamma, x : A, \Delta' \vdash Z \text{ type}}\ \text{ctx-last}}{\Gamma \vdash A \text{ type}}\ \text{(I.H.)}$$

Π : produto (de família de tipos)
ou tipo de funções dependentes

Π-formação

$$\frac{\Gamma, x : A \vdash B(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \text{ type}} \; \Pi\text{-F}$$

$(x:A) \to B(x)$

cong:

$$\frac{\begin{array}{c}\Gamma \vdash A \doteq A' \text{ type} \\ \Gamma, x.A \vdash B(x) \doteq B'(x) \text{ type}\end{array}}{\Gamma \vdash \Pi_{(x.A)} B(x) \doteq \Pi_{(x:A')} B'(x)} \; \begin{array}{l}\Pi\text{-jeq} \\ \Pi\text{-cong}\end{array}$$

Π-introdução

$$\frac{\Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x . b(x) : \Pi_{(x:A)} B(x)} \quad \begin{matrix}\lambda \\ \Pi\text{-I}\end{matrix}$$

cong:

$$\frac{\Gamma, x : A \vdash b(x) \doteq b'(x) : B(x)}{\Gamma \vdash \lambda x\ b(x) \doteq \lambda x.\ b'(x) : \Pi_{(x:A)} B(x)} \quad \begin{matrix}\lambda\text{-jeq} \\ \lambda\text{-cong}\end{matrix}$$

Π-eliminação

$$\frac{\Gamma \vdash f : \Pi_{(x:A)} B(x)}{\Gamma, x : A \vdash f\,x : B(x)} \quad \begin{array}{l} \text{ev} \\ \text{app} \\ \Pi\text{-E} \end{array} \quad *?$$

cong:

$$\frac{\Gamma \vdash f \doteq f' : \Pi_{(x:A)} B(x)}{\Gamma, x : A \vdash f\,x \doteq f'\,x : B(x)} \quad \begin{array}{l} \text{ev-jeq} \\ \text{ev-cong} \end{array}$$

Π-computação

β

$$\frac{\Gamma, x : A \vdash b(x) : B(x)}{\Gamma, x : A \vdash (\lambda y.\, b(y))\, x \doteq b(x) : B(x)}\ \Pi\text{-}\beta$$

η

$$\frac{\Gamma \vdash f : \Pi_{(x:A)} B(x)}{\Gamma \vdash \lambda x.\, f\, x \doteq f : \Pi_{(x:A)} B(x)}\ \Pi\text{-}\eta$$

Definições

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$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n \quad D}{\Gamma \vdash a : A}$$

$D(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ — mesma coisa escrita numa linha

D é uma derivação com raiz $\Gamma \vdash a : A$ com folhas abertas as $\mathcal{H}_1, \ldots, \mathcal{H}_n$.

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n \quad D}{\Gamma \vdash A \text{ type}}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash c := a : A} \ \text{def}[c]\text{-elem}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash C := A \text{ type}} \ \text{def}[C]\text{-type}$$

Isso enriquece nossa teoria de tipos com uma nova constante C e com as regras seguintes:

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{\Gamma \vdash c : A} \ \text{c-elem}$$

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{\Gamma \vdash C \text{ type}} \ \text{c-type}$$

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{\Gamma \vdash c \doteq a : A} \ \text{c-def}$$

$$\frac{\mathcal{H}_1 \quad \cdots \quad \mathcal{H}_n}{\Gamma \vdash C \doteq A \text{ type}} \ \text{C-def}$$

$(\to)$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A \ \mathrm{type} \qquad \Gamma \vdash B \ \mathrm{type}}{\Gamma, x.A \vdash B \ \mathrm{type}}\ \mathrm{kfam}}{\Gamma \vdash \Pi_{(x:A)}.B \ \mathrm{type}}\ \Pi}{\Gamma \vdash A \to B := \Pi_{(x:A)} B \ \mathrm{type}}\ \mathrm{def}\,[\to]\text{-type}$$

$$\dfrac{\Gamma \vdash A \ \mathrm{type} \qquad \Gamma \vdash B \ \mathrm{type}}{\Gamma \vdash A \to B \ \mathrm{type}}\ (\to)\text{-type}\ (\to)\text{-F}$$

$$\dfrac{\Gamma \vdash A \ \mathrm{type} \qquad \Gamma \vdash B \ \mathrm{type}}{\Gamma \vdash A \to B \doteq \Pi_{(x:A)} B \ \mathrm{type}}\ (\to)\text{-def}$$

+ cong

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x : A \vdash b(x) : B}{\Gamma \vdash \lambda x.\, b(x) : A \to B}\ \begin{matrix}\lambda \\ (\to)\text{-I}\end{matrix} \qquad + \text{ cong}$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma, x : A \vdash f\, x : B}\ \begin{matrix}\text{ev} \\ \text{app} \\ (\to)\text{-E}\end{matrix} \qquad + \text{ cong}$$

$$\frac{\Gamma \vdash B \text{ type} \qquad \Gamma, x : A \vdash b(x) : B}{\Gamma, x : A \vdash (\lambda y.\, b(y))\, x \doteq b(x) : B}\ \beta$$

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \lambda x.\, f\, x \doteq f : A \to B}\ \eta$$

identidade

$$\dfrac{\dfrac{\Gamma \vdash A \text{ type} \qquad \dfrac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A}\,\text{gen}(x:A)}{\Gamma \vdash \lambda x.x : A \to A}\,\lambda\,(\to)\text{-I}}{\Gamma \vdash \text{id}_A := \lambda x.x : A \to A}\,\text{def}[\text{id}]\text{-elem}$$

$$\rightsquigarrow \quad \dfrac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \text{id}_A : A \to A}\,\text{id-elem}$$

$$\rightsquigarrow \quad \dfrac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \text{id}_A \doteq \lambda x.x : A \to A}\,\text{id-def}$$

currificação

nem temos (ainda)

$$(\circ) \quad : ((B \to C) \times (A \to B)) \to (A \to C)$$
$$: (B \to C) \to ((A \to B) \to (A \to C))$$

$$\Pi_{(x:A)} \Pi_{(y:B(x))} C(x,y) \qquad (x:A) \to (y:B(x)) \to C(x,y)$$

$$\Pi_{(x,y:A)} C(x,y) \overset{\text{sug}}{\equiv} \Pi_{(x:A)} \Pi_{(y:A)} C(x,y)$$

Composição

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma, f : B^A, x : A \vdash f\,x : B}\,?}{\Gamma, \overset{g\ f}{\ldots}, x : A \vdash f\,x : B}\,\mathbf{w}(\Gamma; g : C^B) \qquad \dfrac{\Gamma \vdash B \text{ type} \qquad \Gamma \vdash C \text{ type}}{\Gamma, \ldots, x : A, y : B \vdash g(y) : C}\,?}{\cdots \vdash \cdots \quad \Gamma, g : C^B, f : B^A, x : A \vdash g\,(f\,x) : C}\,\mathbf{sub}(y := f\,x)\text{-elem}}{\Gamma, \ldots \vdash C^A \text{ type} \qquad \Gamma, g : C^B, f : B^A \vdash \lambda x.\, g\,(f\,x) : C^A}\,\lambda}{\dfrac{A \quad B \quad C}{\Gamma \vdash B^A \to C^A \text{ type}} \qquad \Gamma, g : C^B \vdash \lambda f\ \lambda x.\, g\,(f\,x) : B^A \to C^A}\,\lambda}{\Gamma \vdash \lambda g.\, \lambda f.\, \lambda x\ g\,(f\,x) : C^B \to (B^A \to C^A)}\,\lambda}{\Gamma \vdash \mathrm{comp} := \lambda g.\, \lambda f.\, \lambda x\ g\,(f\,x) : C^B \to (B^A \to C^A)}\,\mathbf{def}[\mathrm{comp}]\text{-elem}$$

Leis de identidade e composição

Associativity

$$(h \circ g) \circ f \doteq h \circ (g \circ f) \quad : \; ?$$

Unit:

$$id_B \circ f \doteq f \quad : \; ?$$

$$f \circ id_A \doteq f \quad : \; ?$$

Constante

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, k : B \vdash \mathrm{const}_k : A \to B}$$ const-elem

$\mathrm{const}_k := ?$

$\mathrm{const}_c \circ f \doteq \mathrm{const}_c : ?$

$g \circ \mathrm{const}_c \doteq \mathrm{const}_{(g\,c)} : ?$

flip ou swap

$$\frac{\begin{array}{c}\Gamma \vdash A \text{ type} \\ \Gamma \vdash B \text{ type} \\ \Gamma, x : A, y : B \vdash C(x,y) \text{ type}\end{array}}{\Gamma \vdash \sigma : \prod_{x:A} \prod_{y:B} C(x,y) \to \prod_{y:B} \prod_{x:A} C(x,y)}$$ flip swap σ

$\sigma \circ \sigma \doteq ? : ?$

$\mathbb{N}$

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formação

$$\frac{}{\vdash \mathbb{N} \text{ type}}\ \mathbb{N}\text{-F}$$

introdução (construtores)

$$\frac{}{\vdash 0_{\mathbb{N}} : \mathbb{N}} \qquad \frac{}{\vdash \mathrm{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}}$$

eliminação (princípio da indução)

$$\frac{\begin{array}{l} \Gamma, n : \mathbb{N} \vdash P(n) \text{ type} \\ \Gamma \vdash b : P(0) \\ \Gamma \vdash s : \prod_{k:\mathbb{N}} \left(P(k) \to P(\mathrm{succ}\ k)\right) \end{array}}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(b, s) : \prod_{n:\mathbb{N}} P(n)}\ \mathbb{N}\text{-ind}$$

$\mathbb{N}$ computação (um para cada construtor)

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$$\frac{\begin{array}{l}\Gamma, n : \mathbb{N} \vdash P(n) \text{ type} \\ \Gamma \vdash p_0 : P(0) \\ \Gamma \vdash p_s : \prod_{k:\mathbb{N}} (P(k) \to P(\text{succ } k))\end{array}}{\begin{array}{l}\Gamma \vdash \text{ind}_{\mathbb{N}}(p_0, p_s)\ 0_{\mathbb{N}} \doteq p_0 : P(0) \\ \Gamma, n : \mathbb{N} \vdash \text{ind}_{\mathbb{N}}(p_0, p_s)\ (\text{succ } n) \doteq p_s\ n\ (\text{ind}(p_0, p_s)\ n) : P(\text{succ } n)\end{array}}\ \mathbb{N}\text{-}\beta$$

η?

Dica sobre o $id_B \circ f \doteq f$:

Basta $\Gamma, x : A \vdash id\ (f\ x) = f\ x : B$

Por quê?

$$\dfrac{\dfrac{\Gamma, x : A \vdash id\ (f\ x) = f\ x : B}{\Gamma \vdash \lambda x.\ id\ (f\ x) \doteq \lambda x.\ f\ x : A \to B}\ \lambda\text{-cong} \qquad \dfrac{\Gamma \vdash f : A \to B}{\Gamma \vdash \lambda x.\ f\ x \doteq f : A \to B}\ \eta}{\Gamma \vdash id \circ f \doteq f : A \to B}\ ?$$

$\mathbb{N}$. adição

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spec:

$$\mathrm{add}_{\mathbb{N}} : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$

$$\mathrm{add}(m, 0) \doteq m$$

$$\mathrm{add}(m, (\mathrm{succ}\ n)) \doteq \mathrm{succ}(\mathrm{add}(m, n))$$

Queremos usar o $\mathrm{ind}_{\mathbb{N}}$:

$$\frac{\Gamma, n : \mathbb{N} \vdash P(n) \text{ type} \quad \Gamma \vdash b : P(0) \quad \Gamma \vdash s : \prod_{k:\mathbb{N}} \left(P(k) \to P(\mathrm{succ}\ k)\right)}{\Gamma \vdash \mathrm{ind}_{\mathbb{N}}(b, s) : \prod_{n:\mathbb{N}} P(n)}\ \mathbb{N}\text{-ind}$$

Como?

$\Gamma := m : \mathbb{N}$ ✓ (FAM) $m : \mathbb{N}, n : \mathbb{N} \vdash \mathbb{N}$ type

$P(n) := \mathbb{N}$ ✓ (BASE) $m : \mathbb{N} \vdash \mathrm{add}_{\mathrm{base}}(m)\ \mathrm{add}_{\mathrm{zero}}(m) : \mathbb{N}$

$b := ?$ ✓ (STEP) $m : \mathbb{N} \vdash \mathrm{add}_{\mathrm{step}}(m)\ \mathrm{add}_{\mathrm{succ}}(m) : \prod_{n:\mathbb{N}} (\mathbb{N} \to \mathbb{N})$

$s := ?$

$$\frac{}{m : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}(b, s) : \prod_{n:\mathbb{N}} \mathbb{N}}\ \mathbb{N}\text{-ind}$$

$$m : \mathbb{N} \vdash \mathrm{add}\ m : \mathbb{N} \to \mathbb{N}$$

(BASE) $m : \mathbb{N} \vdash \mathrm{add}_{\mathrm{zero}}(m) : \mathbb{N}$

$$\frac{\vdots}{m : \mathbb{N} \vdash m : \mathbb{N}}$$

$$\mathrm{add}_{\mathrm{zero}}(m) := m$$

(STEP) $m : \mathbb{N} \vdash \mathrm{add}_{\mathrm{succ}}(m) : \prod_{n:\mathbb{N}} (\mathbb{N} \to \mathbb{N})$

$$\frac{\vdots}{m : \mathbb{N} \vdash \lambda n.\, \mathrm{succ} : \prod_{n:\mathbb{N}} (\mathbb{N} \to \mathbb{N})}$$

$$\mathrm{a}_{\mathrm{s}}(m) := \lambda n.\mathrm{succ}$$

$$\mathrm{add}_{\mathbb{N}}\ m := \mathrm{ind}_{\mathbb{N}}\left(\mathrm{a}_{\mathrm{z}}(m),\ \mathrm{a}_{\mathrm{s}}(m)\right) : \mathbb{N} \to \mathbb{N}$$

spec: $\mathrm{add}\,(m, (\mathrm{succ}\ n)) \doteq \mathrm{succ}\,(\mathrm{add}\,(m, n))$

$$
\begin{aligned}
&\mathrm{add}\ m\ (\mathrm{succ}\ n) \\
&\quad \doteq \mathrm{ind}\,(a_z(m), a_s(m))\ (\mathrm{succ}\ n) \\
&\quad \doteq \mathrm{ind}\,(m, a_s(m))\ (\mathrm{succ}\ n) \\
&\quad \doteq a_s(m)\ n\ (\underbrace{\mathrm{ind}\,(m, a_s(m))}\ n) \\
&\quad \doteq a_s(m)\ n\ (\mathrm{add}\ m\ n) \\
&\quad \doteq \mathrm{succ}\,(\mathrm{add}\ m\ n)
\end{aligned}
$$

HW

a $\mathrm{add}_{\mathbb{N}}$ construída atende a spec

$$\Gamma, n : \mathbb{N} \vdash \mathrm{ind}_{\mathbb{N}}\,(p_0, p_s)\ (\mathrm{succ}\ n) \doteq p_s\ n\ (\mathrm{ind}\,(p_0, p_s)\ n) : P\ (\mathrm{succ}\ n)$$

Pattern matching

$$f\ 0 := p_0$$
$$f\ (\mathrm{succ}\ n) := p_s\ (n, f\ n)$$

$$\mathrm{add}' : \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$$
$$\left\{\begin{array}{l} \mathrm{add}'\ 0\ 0 := 0 \\ \mathrm{add}'\ 0\ (S\ n) := S\ n \\ \mathrm{add}'\ (S\ m)\ 0 := S\ m \\ \mathrm{add}'\ (S\ m)\ (S\ n) := S\ (S\ (\mathrm{add}'\ m\ n)) \end{array}\right\}$$

$$\mathrm{fib} : \mathbb{N} \to \mathbb{N}$$
$$\left\{\begin{array}{l} \mathrm{fib}\ 0 := 0 \\ \mathrm{fib}\ 1 := 1 \\ \mathrm{fib}\ (S\ (S\ n)) := \\ \quad \mathrm{fib}\ (S\ n) + \mathrm{fib}\ n \end{array}\right\}$$

$$\mathrm{ev}, \mathrm{od} : \mathbb{N} \to \mathbb{B}$$
$$\left\{\begin{array}{ll} \mathrm{ev}\ 0 := tt & \mathrm{od}\ 0 := ff \\ \mathrm{ev}\ (S\ n) := \mathrm{od}\ n & \mathrm{od}\ (S\ n) := \mathrm{ev}\ n \end{array}\right\}$$

Cadê a (=) ? [props-as-types]

$$? : 1 + 2 + \cdots + n = n(n+1)/2$$

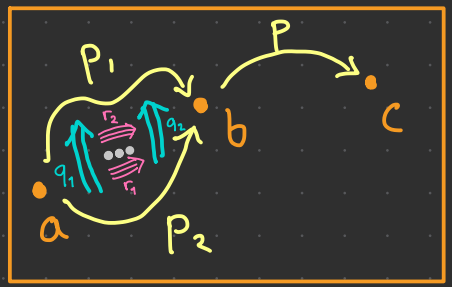
$$\mathrm{Id}_{\mathbb{N}}(1 + 2 + \cdots + n,\ n(n+1)/2)$$

A type, $a : A$, $b : A$

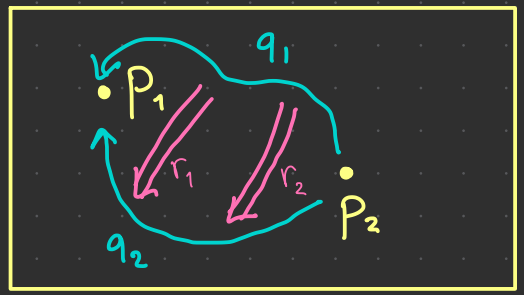
$\vdash \mathrm{Id}_A(a, b)$ type

o tipo de todas a identificações de a com b

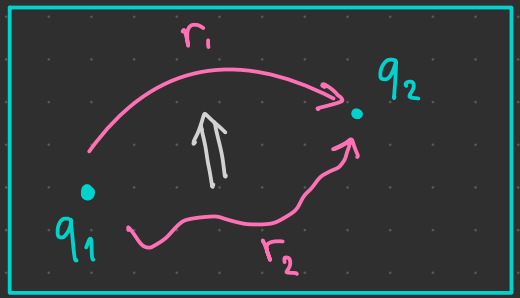
Identificações



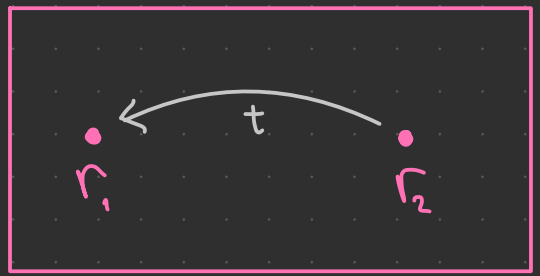
$a : A,\ b : A \vdash \mathrm{Id}_A(a, b)$



$\mathrm{Id}_A(a,b)$



$\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p_2, p_1)$



$\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p_2,p_1)}(q_1, q_2)$

Unit type (1)

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$\star : \mathbb{1}$

$P(\star) = A$

$\mathrm{ind}_{\mathbb{1}} : P(\star) \to \prod_{(x:\mathbb{1})} . P(x)$

$\mathrm{ind}_{\mathbb{1}} : A \to (\mathbb{1} \to A)$

$\mathrm{ind}_{\mathbb{1}}(p, \star) \doteq p$

$x \cdot A$

$f\,\star := p$

$\mathrm{pt}_x := \mathrm{ind}_{\mathbb{1}}\, x : \mathbb{1} \to A$

Empty type $(\mathbb{0})$

$$\mathrm{ind}_{\mathbb{0}} : \prod_{(x:\mathbb{0})} P(x)$$

boom

$$\text{ex-falso} := \mathrm{ind}_{\mathbb{0}} : \mathbb{0} \to A$$

$$\neg A := A \to \mathbb{0}$$

$$\text{is-empty}(A) := A \to \mathbb{0}$$

$$\neg\neg A \neq A$$

$$\equiv \quad \neg(A \to \mathbb{0}) \equiv (A \to \mathbb{0}) \to \mathbb{0}$$

$? \; : \quad (P \to Q) \to (\neg Q \to \neg P)$

$$\lambda f . \lambda \tilde{q} \; \lambda p . \; \underbrace{\tilde{q} \, (f \, p)}_{\mathbb{0}}$$

$f : P \to Q$

$\tilde{q} : Q \to \mathbb{0}$

$p : P$

Sum (coproduct)

$$\mathrm{inl} : A \to A + B$$

$$\mathrm{inr} : B \to A + B$$

$$\mathrm{ind}_+ : \left(\prod_{(x:A)} P(\mathrm{inl}\, x)\right) \to \left(\left(\prod_{y:B} P(\mathrm{inr}\, y)\right) \to \prod_{z:A+B} P(z)\right)$$

$$\mathrm{ind}_+ : (A \to X) \to (B \to X) \to (A + B \to X)$$

$$\mathrm{ind}_+(f, g, \mathrm{inl}\, x) \doteq f\, x \qquad h(\mathrm{inl}\, x) := f\, x$$

$$\underbrace{\mathrm{ind}_+(f, g,}_{[f \mid g]} \mathrm{inr}\, y) \doteq g\, y \qquad h(\mathrm{inr}\, y) := g\, y$$

$$f + g : A + B \to A' + B'$$

$$(f+g)(\mathrm{inl}\ x) := \mathrm{inl}\ (f\ x)$$

$$(f+g)(\mathrm{inr}\ y) := \mathrm{inr}\ (g\ y)$$

A, B types

$$? : \underbrace{\text{is-empty}(B)}_{B \to \mathbb{1}} \to ((A + B) \to A)$$

$$\mathrm{id} =: f : A \to A \qquad \tilde{b} : B \to \mathbb{1}$$

$$\text{ex-falso} \circ \tilde{b} =: g : B \to A$$

$\mathbb{Z}$ $\left(\mathbb{N} \times \mathbb{N} / \sim \text{ não funciona} \right)$

$$\mathbb{Z} := \mathbb{N} + (\mathbb{1} + \mathbb{N})$$

$\cdots$	-4	-3	-2	-1	0	1	2	3	4	$\cdots$
$\cdots$	3	2	1	0	$\star$	0	1	2	3	$\cdots$

$$\text{in-pos} := \text{inr} \circ \text{inr} : \mathbb{N} \to \mathbb{Z}$$

$$\text{in-neg} := \text{inl} : \mathbb{N} \to \mathbb{Z}$$

$$-1_{\mathbb{Z}} := \text{in-neg}\ 0_{\mathbb{N}}$$

$$0_{\mathbb{Z}} := \text{inl}\ (\text{inr}\ \star)$$

$$1_{\mathbb{Z}} := \text{in-pos}\ 0_{\mathbb{N}}$$

$$f\ (-1_{\mathbb{Z}}) := p_{-1}$$

$$f\ (0_{\mathbb{Z}}) := p_0$$

$$f\ (1_{\mathbb{Z}}) := p_1$$

$$f\ (\text{in-neg}\ (\text{succ}_{\mathbb{N}}\ n)) := p_{-s}\ (n, f\ (\text{in-neg}\ n))$$

$$f\ (\text{in-pos}\ (\text{succ}_{\mathbb{N}}\ n)) := p_{s}\ (n, f\ (\text{in-pos}\ n))$$

$$
\begin{aligned}
\text{succ}\ (-1) &:= 0 \\
\text{succ}\ 0 &:= 1 \\
\text{succ}\ 1 &:= \text{in-pos}\ (1) \\
\text{succ}\ (\text{in-neg}\ (\text{succ}\ n)) &:= \text{in-neg}\ n \\
\text{succ}\ (\text{in-pos}\ (\text{succ}\ n)) &:= \text{in-pos}\ (\text{succ}\ (\text{succ}\ n))
\end{aligned}
$$

Σ

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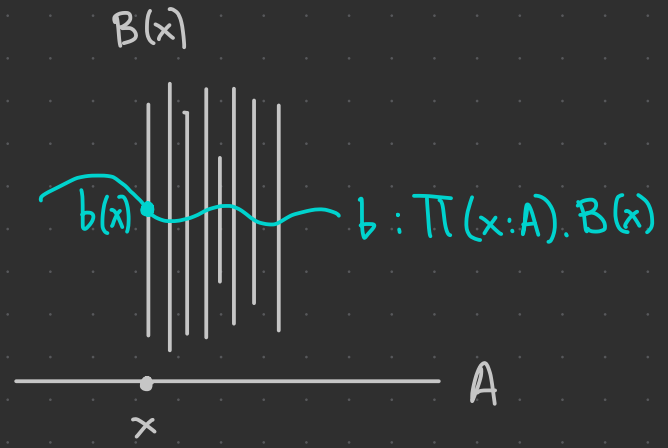
$$\frac{\Gamma, a : A \vdash B(a) \text{ type}}{\Gamma \vdash \Sigma(a:A).\, B(a) \text{ type}}$$

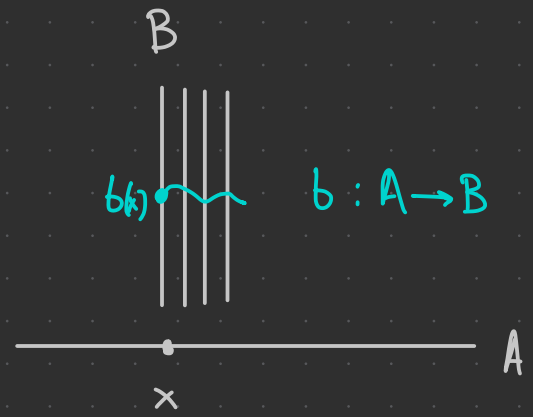
$B : A \to \mathcal{U}$ ← universo Type

$$(x:A) \to B(x) \to \big((y:A) \times B(y)\big)$$

$$\text{pair} : \Pi(x:A).\, \underbrace{\big(B(x) \to \Sigma(y:A).\, B(y)\big)}_{\Pi(_ : B(x)).\, \Sigma(y:A).\, B(y)}$$

Π





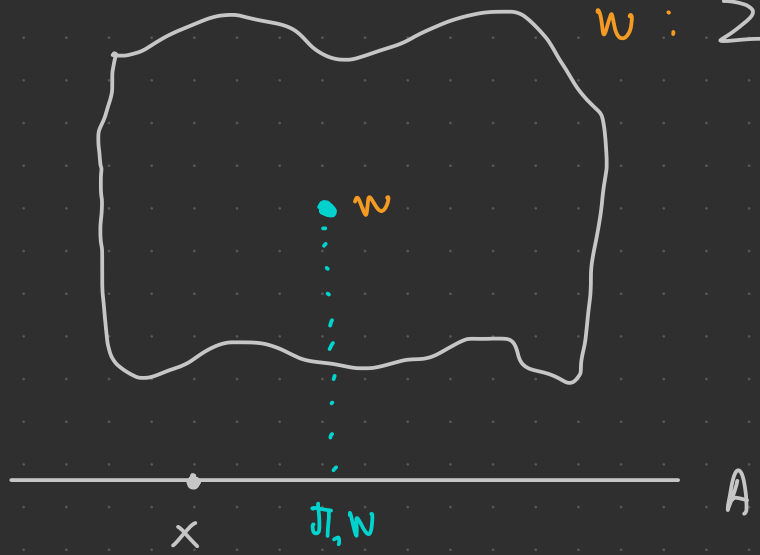
$\Pi(x:A).\ B(x)$

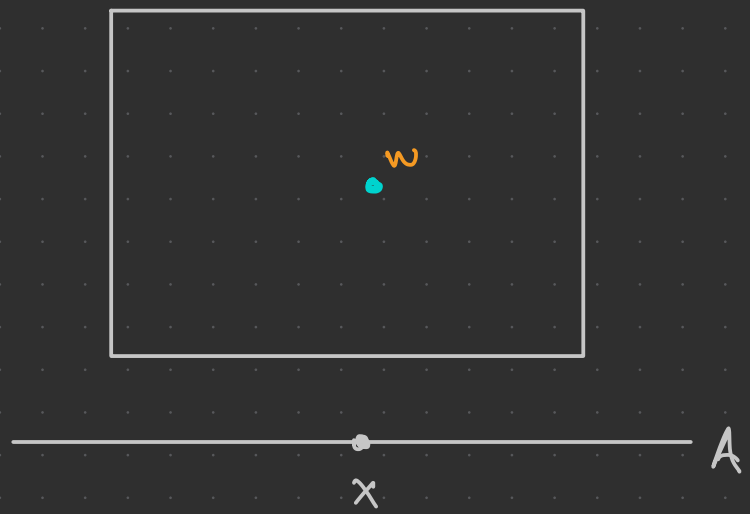
$\lambda(x:A).\ b(x)$

$A \to B$

Σ

$$w : \Sigma(x:A).\,B(x) \qquad \underbrace{(a:A,\ b:B(a))}_{(x:A)\times B(x)}$$





$$\pi_1 : \Sigma(x:A).\,B(x) \to A \qquad \pi_1 : A \times B \to A$$

$$\pi_2 : \Pi\big(w : \Sigma(x:A).\,B(x)\big).\,B(\pi_1 w) \qquad \pi_2 : A \times B \to B$$

$$(w : (x:A)\times B(x)) \to B(\pi_1 w)$$

$$\mathrm{ind}_\Sigma : \left(\prod(x:A)(y:B(x)).\ P(\mathrm{pair}(x,y))\right) \to \left(\prod(z:\Sigma(x:A)B(x)).\ P(z)\right)$$

$$\mathrm{ind}_\Sigma\ (g,\ \mathrm{pair}(x,y)) \doteq g(x,y)$$

$$f\ (\mathrm{pair}(x,y)) := g(x,y)$$

$$\pi_1 : (\Sigma(x:A).\ B(x)) \to A$$

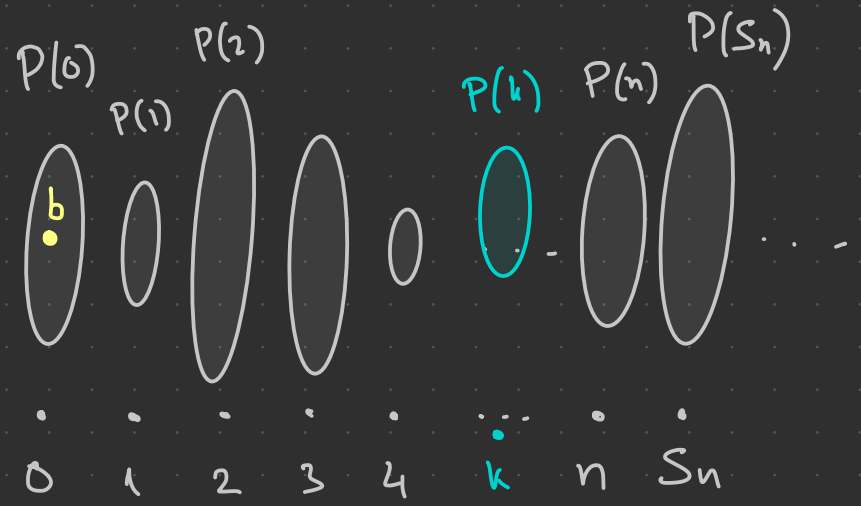
$$\pi_1\ (x,y) := x$$

$$\pi_2\ (x,y) := y$$

- $\mathbb{N} : \mathcal{U}$
- cons

  $0_{\mathbb{N}} : \mathbb{N}$

  se $n : \mathbb{N}$ então $S\,n : \mathbb{N}$
- ind/rec



Dados:

- $P : \mathbb{N} \to \mathcal{U}$
- $b : P(0)$
- $f : \Pi(n:\mathbb{N}).\ P(n) \to P(Sn)$
- $k : \mathbb{N}$

(ob)temos: $\mathrm{ind}_{\mathbb{N}}(P, b, f, k) : P(k)$

$\mathrm{ind}_{\mathbb{N}}(P, b, f, 0) \doteq b \quad : P(0)$

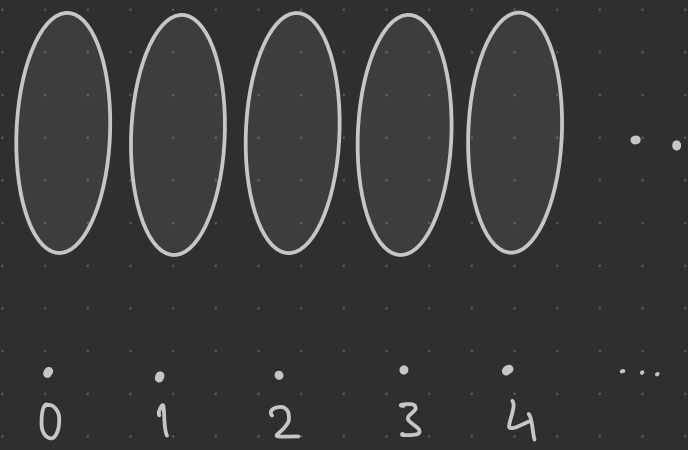
$\mathrm{ind}_{\mathbb{N}}(P, b, f, Sm) \doteq$

$f\,m\,(\mathrm{ind}_{\mathbb{N}}(P, b, f, m))$

$: P(Sm)$

- rec/ind
  - $A : \mathcal{U}$
  - $b : A$
  - $f : \mathbb{N} \to A \to A$

$P(n) := A$



$$\mathrm{rec}(A, b, f, k) : A$$

$$\equiv \mathrm{ind}(\lambda_.A, b, f, k)$$

Identidades

⚠ Id_A vs id_A

- formação:

$A : \mathcal{U}, \quad a : A, \quad b : A$ $\qquad \mathbb{N}, \quad 0, \quad s0$

$\rightsquigarrow \quad \mathrm{Id}_A(a,b) : \mathcal{U}$

$\quad a =_A b : \mathcal{U}$

as identificações de a com b
caminhos

- construtor

$A : \mathcal{U}$

$a : A$

$\mathrm{refl}_A(a) : \mathrm{Id}_A(a, a)$

$\mathrm{idpath}_A(a) : a =_A a$

- indução

(cliff-hanger)

Espaços métricos

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$(X ; d)$

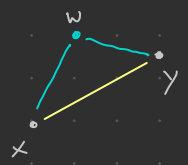
$X : \mathrm{Set}\ \alpha$

$d : X \times X \to \mathbb{R}$ Distance

+ leis: d é uma métrica ou distância:

pré-métrica ou pseudo métrica

$$\begin{cases} d(x,y) \geq 0 \\ d(x,x) = 0 \qquad (d(x,y) = 0 \Leftarrow x = y) \\ d(x,y) = d(y,x) \\ d(x,y) \leq d(x,w) + d(w,y) \qquad [\text{triangular}] \\ d(x,y) = 0 \Rightarrow x = y \end{cases}$$

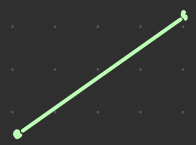


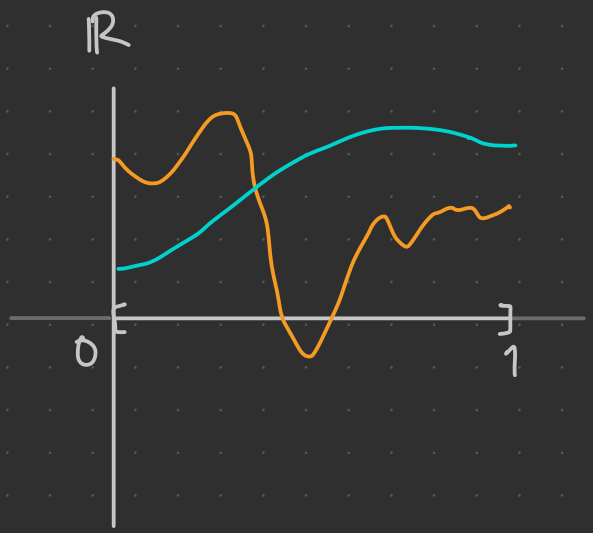
discreta/o

$$d(x,y) \overset{\text{def}}{\equiv} \begin{cases} 0, & \text{caso } x = y \\ 1, & \text{c.c.} \end{cases}$$

$\mathbb{R}$ $\quad \mathbb{R}^2 \quad \mathbb{R}^3 \quad \cdots \quad \mathbb{R}^n \quad \cdots$ euclideanos $\qquad (A\,;\,d)$

$$f : \underbrace{[0,1] \xrightarrow{cts} \mathbb{R}}_{X}$$





norma

Size

$$\|\cdot\| : X \longrightarrow \mathbb{R}$$

$$\|f\| = \int_0^1 |fx|\,dx$$

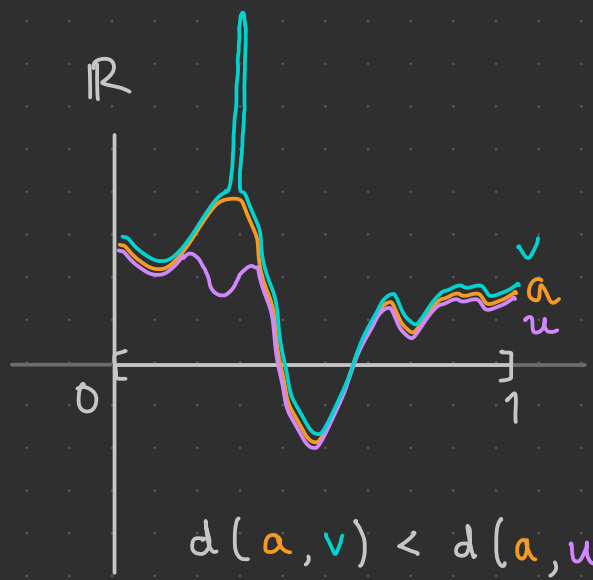
$$d(f,g) \overset{\text{def}}{\equiv} \|f-g\| = \int_0^1 |fx - gx|\,dx$$

$$B_\varepsilon(a) \overset{\text{def}}{\equiv} \{x : X \mid d(x,a) < \varepsilon\}$$

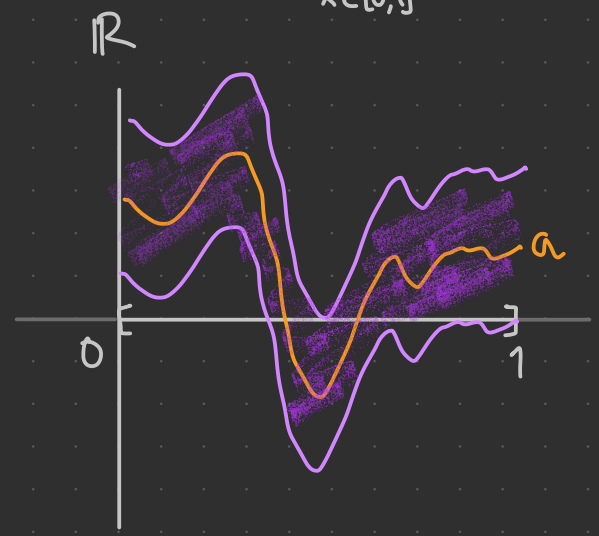
$$\|f\| = \int_0^1 |fx|\,dx$$

$$\|f\| = \sup_{x\in[0,1]} |fx|$$

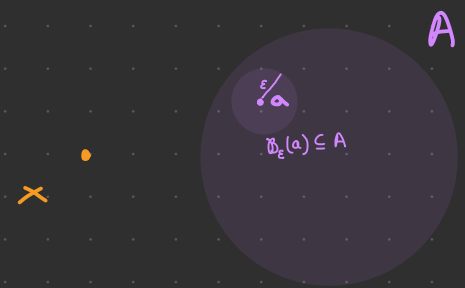
$$d(f,g) = \|f-g\| = \sup_{x\in[0,1]} |fx - gx|$$



$d(a,v) < d(a,u)$



$$d(x, A) \overset{\text{def}}{\equiv} \inf\{d(x,a) \mid a \in A\}$$



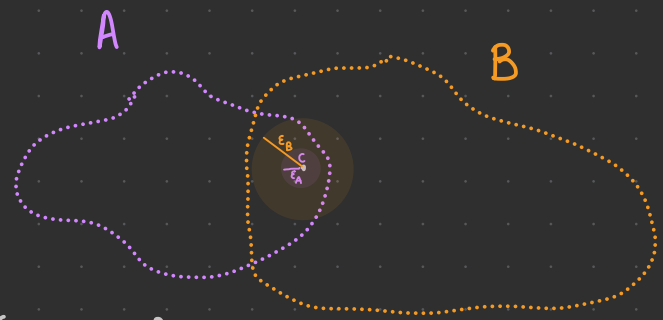
$$\operatorname{diam}(A) \overset{\text{def}}{\equiv} \sup\{d(a,b) \mid a, b \in A\}$$

$$A \text{ open} \overset{\text{def}}{\iff} (\forall a)(\exists \varepsilon > 0)[\mathcal{B}_\varepsilon(a) \subseteq A]$$

$$A \text{ closed} \iff X \setminus A \text{ open} \overset{\text{def}}{\iff} ?$$

$$A \text{ clopen} \overset{\text{def}}{\iff} A \text{ open } \& \ A \text{ closed}$$

A, B open



$A \cup B$

$A \cap B$

Esc $\varepsilon := \min\{\varepsilon_A, \varepsilon_B\}$

Preciso: $\mathbb{B}_\varepsilon(c) \subseteq A \cap B$

$\bigcap$?

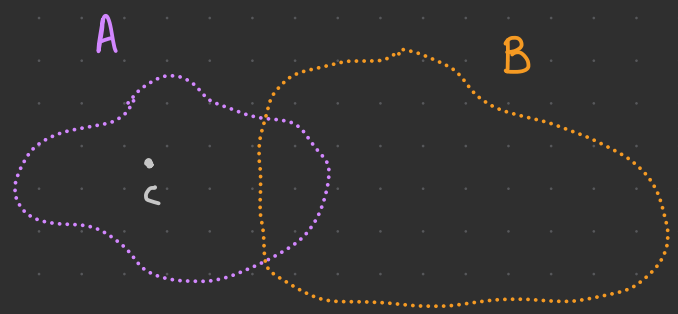
$c \in \bigcap \mathcal{A}$

Esc $\varepsilon := \inf\{\varepsilon_A \mid A \in \mathcal{A}\}$

Seja $x \in \mathbb{B}_\varepsilon(c)$, ou seja: $d(x,c) < \varepsilon$

ALVO: $x \in A \cap B$

$$\dfrac{\dfrac{\dfrac{d(x,c) < \varepsilon \leq \varepsilon_A}{d(x,c) < \varepsilon_A}}{x \in \mathbb{B}_{\varepsilon_A}(c)} \qquad \mathbb{B}_{\varepsilon_A}(c) \subseteq A}{x \in A}$$



$c \in A \cup B$

Caso $c \in A$:

$\vdots$

$\mathcal{B}_{\varepsilon_A}(c) \subseteq A \subseteq A \cup B$

Caso $c \in B$:

similar

$\bigcup$?

Seja $c \in \bigcup \mathcal{A}$.

Logo seja $A_c \in \mathcal{A}$ t.q. $c \in A_c$.

Logo seja $\varepsilon_A > 0$ t.q.

$\mathcal{B}_{\varepsilon_A}(c) \subseteq A_c \subseteq \bigcup \mathcal{A}$.

Seqüências

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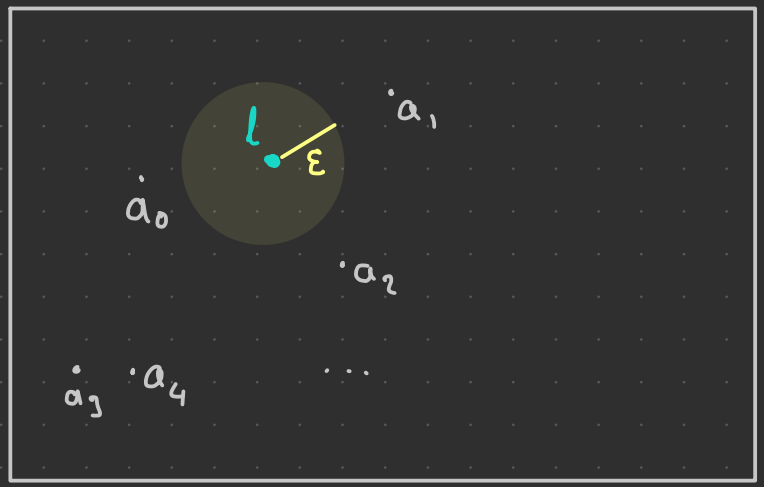
$(a_n)_n$: Seq X $\quad \{a_n\}_{n=0}^{\infty}$ $\quad \left(\frac{1}{n}\right)_{n=5}^{\infty} = \left(\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \dots\right)$

$(a_n \mid n \in \mathbb{N} \in \omega)$: Seq X $\rightsquigarrow$ $\{a_n \mid n \in \mathbb{N}\}$: Set X



Limites

$(a_n)_n \to \ell \overset{def}{\iff} (\forall \varepsilon > 0)[\text{ eventualmente } (a_n)_n \subseteq \mathcal{B}_\varepsilon(\ell)]$

$\ell = \lim_n a_n$ $\qquad (\exists N)[(a_n)_{n \geq N} \subseteq \mathcal{B}_\varepsilon(\ell)]$

$\ell = \lim (a_n)_n$ $\qquad (\exists N)(\forall i \geq N)[\ a_i \in \mathcal{B}_\varepsilon(\ell)]$

$[\ d(a_i, \ell) < \varepsilon]$

$\lim_X : \text{Seq } X \to X$ **ou** $_ = \lim _ \ : X \times \text{Seq } X \to \text{Prop}$

Precisamos justificar a notação com $(=)$:

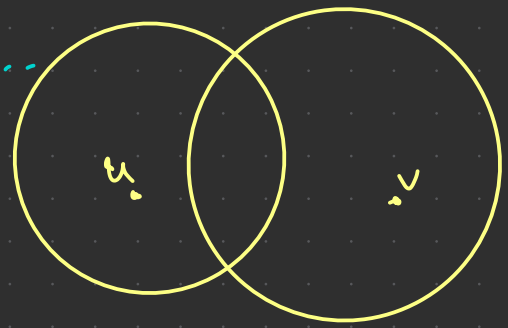
Θ (Unicidade de limites)

h_u: $u = \lim (a_n)_n \iff (\forall \varepsilon > 0)(\exists N) \ldots$

h_v: $v = \lim (a_n)_n \iff (\forall \varepsilon > 0)(\exists N) \ldots$

$\vdash u =_x v$

$(\forall \varepsilon > 0)(x < \varepsilon) \implies x = 0)$



Basta mostrar: $(\forall \varepsilon > 0)[\, d(u,v) < \varepsilon \,]$

Seja $\varepsilon > 0$. [ataquei o ALVO] — Alvo: $d(u,v) < \varepsilon$

Seja N_u t.q. $(\forall i \geq N_u)[\, d(a_i, u) < \varepsilon \,]$ h_{N_u} $[\, h_{n_u}\ \varepsilon := \varepsilon_u \,]$

Seja N_v t.q. $(\forall i \geq N_v)[\, d(a_i, v) < \varepsilon \,]$ h_{N_v} $[\, h_{v_w}\ \varepsilon := \varepsilon_v \,]$

Seja $N \geq \{N_u, N_v\}$. [?] — $[\, N := \max\{N_u, N_v\} \,]$

Logo $\quad d(a_N, u) < \varepsilon_u \overset{h_{\varepsilon_u}}{\vdots} \quad [\ h_{N_u} \quad 1 := N\ ]$

$\quad d(a_N, v) < \varepsilon_v \overset{h_{\varepsilon_u}}{\vdots} \quad [\ \ldots\ ]$

Calc:

$$
\begin{aligned}
& d(u,v) \\
& \leq d(u, a_N) + d(a_N, v) && [\text{triang}] \\
& = d(a_N, u) + d(a_N, v) && [\text{sym.}] \\
& < \varepsilon_u + \varepsilon_v && [\ h_{\varepsilon_u}\ ;\ h_{\varepsilon_v}\ ] \\
& < \varepsilon && [\text{esc. de } \varepsilon_u\ ;\ \text{esc. de } \varepsilon_v\ ]
\end{aligned}
$$

Vocabulário

N aberto & $a \in N$

$\Updownarrow$?

N é vizinhança de a $\overset{\text{def}}{\Longleftrightarrow}$ $(\exists \varepsilon > 0)[\, \mathbb{B}_\varepsilon(a) \subseteq N \,]$

$\Updownarrow$?

$(\exists U \text{ aberto})[\, U \subseteq N \;\&\; a \in U \,]$

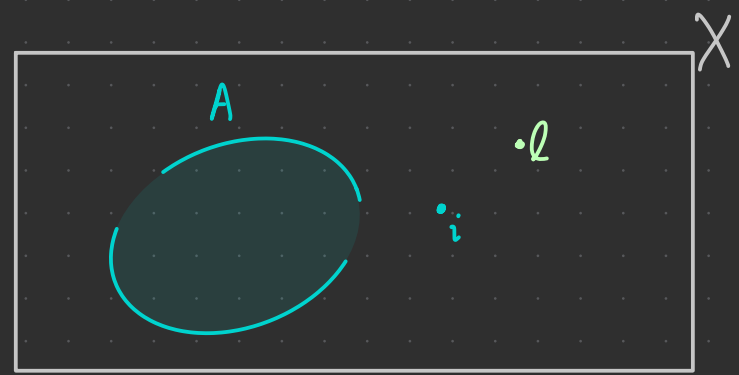
$\mathbb{B}_\varepsilon(a) \setminus \{a\}$: bola puncturada [perfurada?]

ponto de acumulação
ou
ℓ limit point de A

$\Updownarrow$ def

$X \times \mathrm{Set}\, X \to \mathrm{Prop}$

$(\exists (a_n)_n \subseteq A \setminus \{\ell\})[\, (a_n)_n \to \ell \,]$



derived set $\rightsquigarrow$ $A' \overset{\text{def}}{\equiv} \{$ os limit points de $A \}$

Produtos

$$\left(X_1 \times \cdots \times X_n \; ; \; \begin{cases} \sum d_i \\ \max d_i \end{cases} \right)$$

(quais as setas?)

Quem merece o nome?

Intervalos

$[\ \ )$
$(\ \ ]$
$[\ \ )$

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$I \subseteq \mathbb{R}$ intervalo $\overset{\text{def}}{\iff} (\exists a, b \in \overbrace{\mathbb{R} \cup \{\pm\infty\}}^{\text{os reais estendidos}})[\, I = (a,b)\,]$

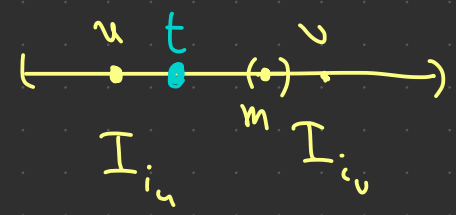
$\iff (\forall u, v \in I)(\forall u \leq t \leq v)[\, t \in I\,]$

Θ. $\bigcap_i I_i$ hab $\Rightarrow$ $\bigcup_i I_i$ intervalo -- Alvo: $t \in \bigcup_i I_i$

Sejam $u, v \in \bigcup_i I_i$, t t.q. $u \leq t \leq v$. (achar i t.q. $t \in I_i$)

Logo sejam i_u, i_v t.q. $u \in I_{i_u}$, $v \in I_{i_v}$

Seja $m \in \bigcap_i I_i$. Logo $m \in I_{i_u}, I_{i_v}$.



Caso $t \leq m$:

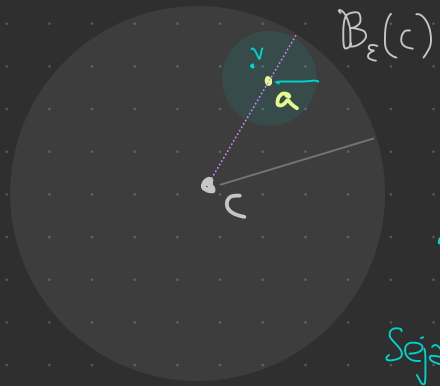
Esc i_u:

$$\frac{u \leq m \in I_{i_u} \qquad u \leq t \leq m}{t \in I_{i_u}}\ I_{i_u} \text{ int}$$

Caso $m < t$:

Esc i_v: ...

Abertos



$\varepsilon_a := \varepsilon - d(a,c)$

$\mathbb{B}_{\varepsilon_a}(a) \subseteq \mathbb{B}_\varepsilon(c)$

Seja $v \in$, ou seja: $d(v,a) < \varepsilon_a$.

Alvo: $d(v,c) < \varepsilon$.

$d(v,c) \leq d(v,a) + d(a,c)$

$< \varepsilon_a + (\varepsilon - \varepsilon_a)$

$= \varepsilon$

- $\emptyset$, X abertos
- $\mathbb{B}_\varepsilon(c)$ aberto
- $\bigcup_\alpha G_\alpha$ aberto ; $\bigcap_{i<n} G_i$ aberto
- G aberto $\Rightarrow$ G é união de bolas abertas

$\Updownarrow$

$(\forall g \in G)(\exists \varepsilon_g)[\mathbb{B}_{\varepsilon_g}(g) \subseteq G]$

$\bigcup_{g \in G} \mathbb{B}_{\varepsilon_g}(g) \subseteq G$

$\supseteq$

- $G \subseteq \mathbb{R}$ aberto $\Rightarrow$ G é união contável de intervalos abertos disjuntos 2-a-2.

Rascunho:

- Considere a $I_{\square} : G \to \wp\mathbb{R}$

  definida pela $I_g \equiv \bigcup \{ I \text{ intervalo} \mid g \in I \subseteq G \}$

- Tentativa de testemunha: $\{ I_g \}_{g \in G}$.
  - É contável?
  - Seus membros são disjuntos dois-a-dois?

DICA: $\mathbb{R}_{\mathbb{Q}}$ contável

(■)

Fechados

- $\emptyset$, X fechados
- F fechado $\Leftrightarrow$ F^c aberto
- $B_\varepsilon[c]$ fechado
- $\bigcap_\alpha F_\alpha$, $\bigcup_{i<n} F_i$ fechado

Interior, Closure, Boundary

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os pontos interiores de A

int(A) $\quad A^\circ \overset{def}{\equiv} \{a \mid (\exists \varepsilon > 0)[B_\varepsilon(a) \subseteq A]\}$

cl(A) $\overline{A}$ $\quad A^- \overset{def}{\equiv} A \cup A'$

bdry(A) $\quad \partial A \overset{def}{\equiv} \{x \mid (\forall \varepsilon > 0)[B_\varepsilon(x)$ interseta ambos os $A, A^c\}$

os pontos de fronteira de A

- A° é o melhor subconjunto aberto de A
- A aberto $\Leftrightarrow A = A^\circ$
- $A^\circ = \bigcup\{G \subseteq A \mid G$ aberto$\}$
- $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ [ache contraexemplo para $(\supseteq)$ no $\mathbb{R}$]
- $A^\circ \cap B^\circ = (A \cap B)^\circ$

$(0,1]^\circ \quad [1,2)^\circ \quad (0,1] \cup [1,2)$

$(0,1) \cup (1,2) \quad (0,2)^\circ$

Θ. A^- é o melhor sup^{er}conjunto fechado de A

Θ. A fechado $\Leftrightarrow$ $A = A^-$

Θ. $A^- = \bigcap \{ F \supseteq A \mid F \text{ fechado} \}$

Θ. $\partial A = A^- \cap \overline{A^c}$

Θ. ∂A fechado

Θ. A fechado $\Leftrightarrow$ $A \supseteq \partial A$

Θ. $\partial(A^c) = ?$ ∂A

Θ. $(\bar{A})^c = ?$ $(A^c)^\circ$

Θ. $\{ x \mid d(x, A) = 0 \} = ?$

os "bem-dentro" os da fronteira

Θ. $A \subseteq X \vdash X = A^\circ \cup (A^c)^\circ \cup \partial A$

os "bem fora"

Θ? $\overline{\mathbb{B}_\varepsilon(c)} \overset{?}{=} \mathbb{B}_\varepsilon[c]$

x. $\overline{A}$, $A^{\circ} = ?$

$A := \mathbb{R}_{\mathbb{Z}}$, $(0, +\infty)$, $(-1,0) \cup (0,1)$, $\mathbb{R}_{\mathbb{Q}}$, $(0, 1]$

x. Pode acontecer...: $\partial A = \emptyset$? $\partial A = X$?

a ponto isolado de A $\overset{def}{\Longleftrightarrow}$ $a \in A \setminus A'$.

Θ. $a \in A$ ponto isolado de A $\Leftrightarrow$ $(\exists \varepsilon > 0)[\, B_{\varepsilon}(a) \setminus \{a\} = \emptyset \,]$

Θ. $A \subseteq \mathbb{R}$ ⊢ A tem no máximo quantidade contável de isolados

Corolário: qualquer incontável $A \subseteq \mathbb{R}$ possui limit point.

D denso $\overset{def}{\Longleftrightarrow}$ $\overline{D} = X$

(everywhere dense)

Funções continuas

f cont. no c $\overset{def}{\iff}$ $(\forall B_\varepsilon(f\,c))(\exists B_\delta(c))[\,f[\,B_\delta(c)] \subseteq B_\varepsilon(f\,c)]$

f cont. $\overset{def}{\iff}$ $(\forall c)[\,f$ cont no $c\,]$

$B_\delta(c) \subseteq f^{-1}[B_\varepsilon(f\,c)]$

Θ. O.s.s.e:

FMC1 (IDMb) $\boxed{\text{(i)}}$ f cont

$\boxed{\text{(ii)}}$ $(\forall (c_n)_n \to c)[\,(f\,c_n)_n \to f\,c\,]$

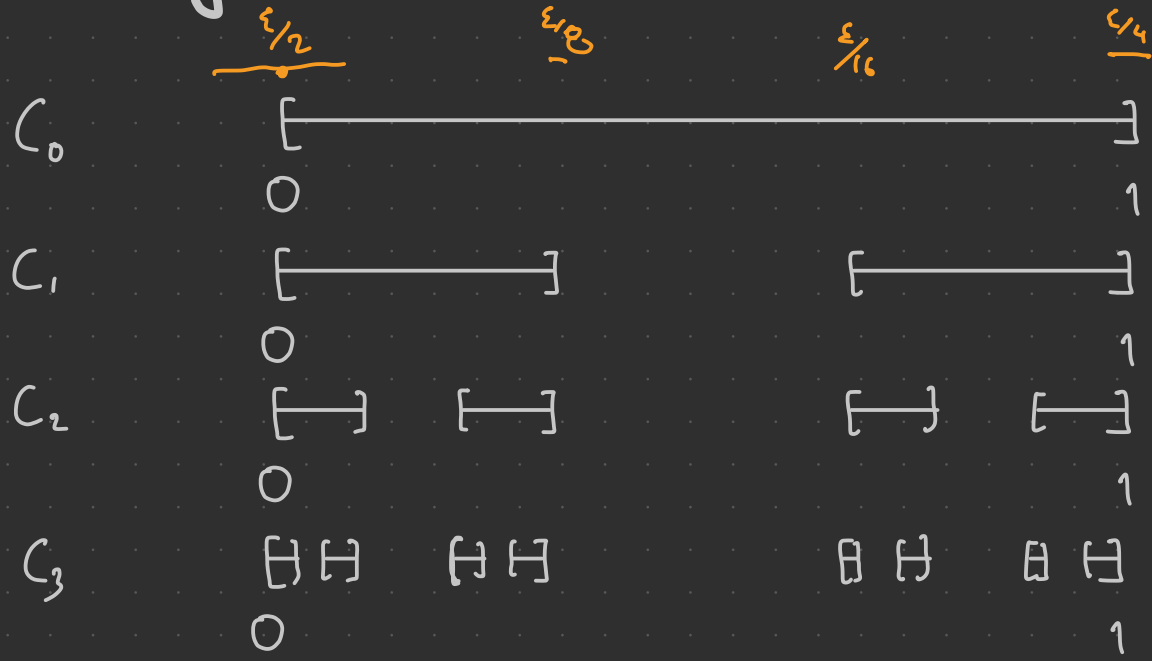
FMC2 (CFR1) $\boxed{\text{(iii)}}$ $(\forall G \subseteq Y$ aberto$)[\,f^{-1}[G]$ aberto$\,]$

$\boxed{\text{(iv)}}$ $(\forall F \subseteq Y$ fechado$)[\,f^{-1}[F]$ aberto$\,]$

Conjuntos fechados no $\mathbb{R}$

lec24

2025-06-25

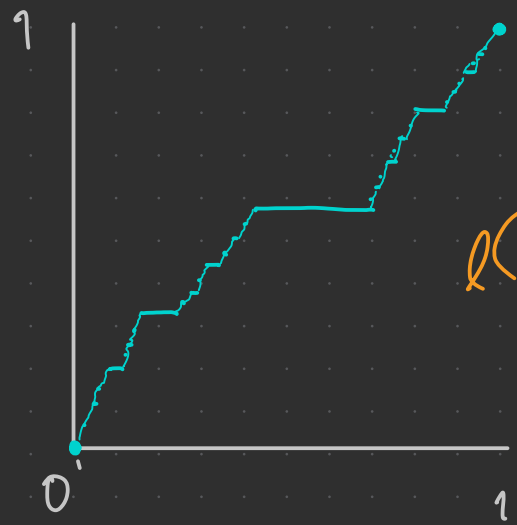


$$C \overset{\text{def}}{=} \bigcap_n C_n$$

$$\cancel{0}, \cancel{1}, \cancel{\tfrac{1}{3}}, \tfrac{2}{3}, \tfrac{1}{9}, \tfrac{2}{9}, \tfrac{4}{9}, \tfrac{5}{9}, \tfrac{7}{9}, \tfrac{8}{9}, \dots$$

hw: $\frac{1}{4} \in C$ $\qquad \frac{\varepsilon}{2}$

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = 1 \qquad (\text{calc. } 2[?])$$



$$\ell(\text{—} \quad \text{— - - -} \quad \text{—}) =$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \cdots$$

$$= \varepsilon \underbrace{\left(1/2 + \frac{1}{4} + \cdots \right)}_{1}$$

$$= \varepsilon$$

$$P_r(E) \in [0, 1]$$

$$= 0$$

$$(\forall \varepsilon > 0)[\, P < \varepsilon \,]$$

Completude

$(x_n)_n$ conv

$\Downarrow$

$(\forall \varepsilon > 0)(\exists N)(\forall u, v \geq N)[\, d(x_u, x_v) < \varepsilon \,]$

Seja ℓ t.q. $(x_n)_n \to \ell$.

Seja $\varepsilon > 0$.

Seja N t.q. $(\forall t \geq N)[\, d(x_t, \ell) < \varepsilon' \,]$.

Fsc. N.

Sejam $u, v \geq N$.

$$\begin{aligned} d(x_u, x_v) &\leq d(x_u, \ell) + d(\ell, x_v) \\ &< \varepsilon' + \varepsilon' \\ &= \varepsilon \end{aligned}$$

Conexos

M desconexo $\iff$ $(\exists A, B$ disjuntos fechados$)(M = A \cup B)$

A, B desconexão de M

Θ. M conexo $\iff$ $\emptyset, M$ são os únicos clopens

$(\Rightarrow)$

Seja A clopen t.q. $A \neq \emptyset, M$.

A, A^c é uma desconexão.

$(\Leftarrow)$ ✓

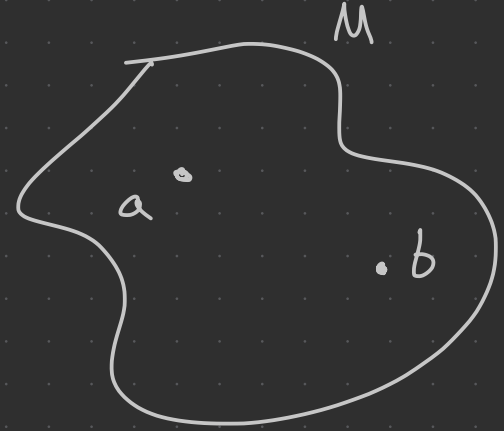
M discon $\iff (\exists f : M \xrightarrow{cts} \{0,1\})[\, f$ sobre$\,]$

$$f\,x = \begin{cases} 0, & x \in A \\ 1, & x \in B \end{cases}$$

$f^{-1}(\{0\}) = A$ aberto

path-connected $\overset{?}{\Longrightarrow}$ connected.

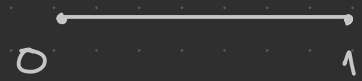
$\overset{?}{\Longleftarrow}$



$$[0,1] \xrightarrow{\text{cts}} M$$

t.q. $f\,0 = a$

$f\,1 = b$



A path-connected $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall a,b \in A)(\exists p)($ p caminho de a para b $)$

p-c $\Longrightarrow$ c

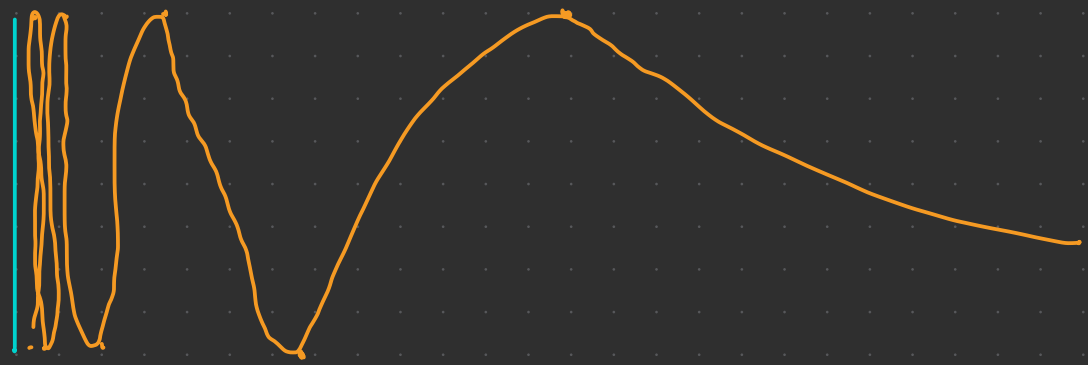
$\Longleftarrow$

4 A $\not\Longleftrightarrow$

5 B $\Longrightarrow$

4 C $\Longleftarrow$

D nenhuma



Compacidade

lec25

2025-06-30

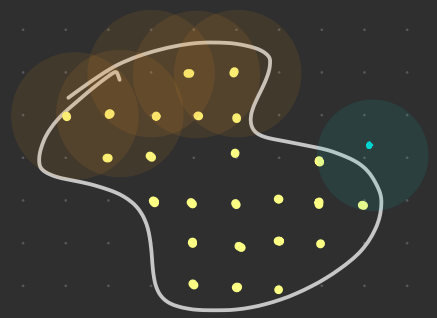
compacto : totally bounded & completo

$\in X$? $\in A$? Não importa. (HW)

A totally bounded $\overset{\text{def}}{\iff}$ $(\forall \varepsilon > 0)(\exists x_1, \dots, x_n)[\{x_1, \dots, x_n\}$ é um ε-net de $A]$

$\Updownarrow$

$\bigcup_i B_\varepsilon(x_i) \supseteq A$



$\ominus$ A tot. bounded $\overset{\not\Leftarrow}{\Longrightarrow}$ A bounded $(\exists c \in A)(\exists \varepsilon > 0)[\mathcal{B}_\varepsilon(c) \supseteq A]$

Sejam n, $x_1, \dots, x_n$ t.q 1-net de A.

Seja $a \in A$.

$$c := x_1$$

$$\varepsilon := \left(\max_{1 \leq i \leq n} d(x_1, x_i) \right) + 1$$

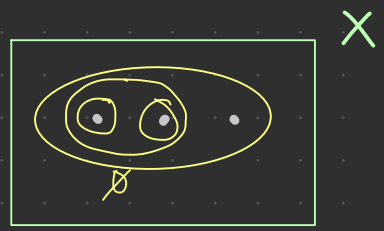
$$a \in \mathcal{B}_\varepsilon(c)$$

$$d(a, c) < \varepsilon$$

Espaços topológicos

$X \equiv (X ; \tau)$ — τ: topologia — coleção de subconjuntos de T

X : Set α (ou Type)



$\tau \subseteq \wp T$

leis

- τ é fechado sob uniões arbitrárias
- τ é fechado sob interseções finitas

τ finer τ' $\overset{\text{def}}{\Longleftrightarrow}$ $\tau \subseteq \tau'$

τ' coarser τ $\overset{\text{def}}{\Longleftrightarrow}$ $\tau \subseteq \tau'$

$F \subseteq X$

F fechado $\overset{def}{\iff}$ $X \setminus F$ aberto

X conexo $\overset{def}{\iff}$ $\emptyset, X$ os únicos clopen

$f : (X, \tau) \xrightarrow{cts} (X'; \tau') \overset{def}{\iff} (\forall U' \in \tau')[\, f^{-1}[U'] \in \tau \,]$

f open $\overset{def}{\iff} (\forall U \subseteq X)[\, f[U] \subseteq X' \,]$

$\subseteq$: subconjunto aberto

$\sqsubseteq$: subconjunto fechado

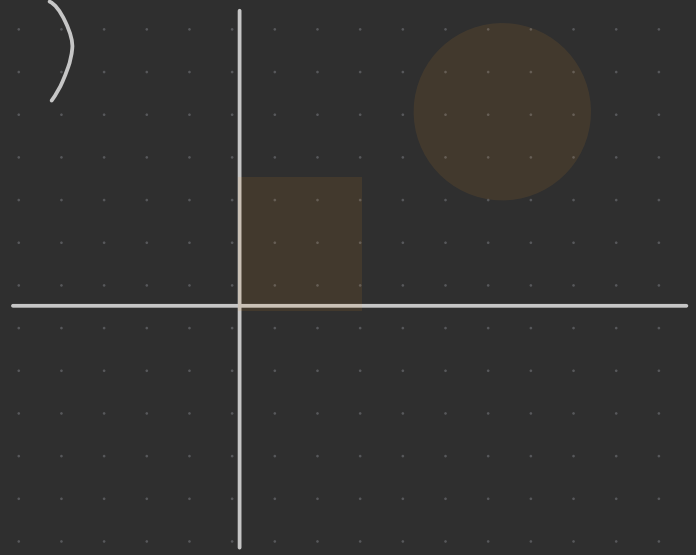
HW $\underline{\mathrm{Top}}$ é uma cat.

Obj: espaços topológicos

Arr: funções continuas

Produtos

$(X \times Y \; ; \; ?)$



Base

$\mathcal{B}$ base para uma topologia no X $\overset{def}{\Longleftrightarrow}$

(i) $(\forall x \in X)(\exists B \in \mathcal{B})[\, x \in B \,]$

(ii) $(\forall x \in B_1 \cap B_2)(\exists B \subseteq B_1 \cap B_2)[\, x \in B \,]$