

Teoria dos tipos Formalização de matemática
Fundamentos de matemática

lec01
2025-03-17

Teoria dos conjuntos

λ -calculus

Teoria das categorias

Programação funcional

Matemática construtiva

Proof assistants

Teoria das demonstrações

Theorem provers

Lógica matemática (FOL, ...)

Static-vs-dynamic

Teoria dos grupos, anéis, ...

axiomas - vs - leis

Teoria dos inteiros, reais, ...

Zermelo - Euclides

Geometria euclideana

Representando números como conjuntos

Zermelo

von Neumann

$$0 \equiv \{\} \equiv \emptyset$$

$$\{\}$$

$$1 \equiv \{\{\}\}$$

$$\{0\}$$

$$2 \equiv \{\{\{\}\}\}$$

$$\{0, 1\}$$

$\vdots$

$\vdots$

$$0 \in 2 ?$$

$$n \mid m \stackrel{\text{def}}{\iff} (\exists k \in \mathbb{Z})(n \cdot k = m)$$

$$A \subseteq B \stackrel{\text{def}}{\iff} (\forall x)(x \in A \Rightarrow x \in B)$$

$3 \subseteq 5 ?$ — essas perguntas deveriam
mesmo ser perguntáveis?

OK sobre os $0, 1, 2, \dots$ mas cadê o $\mathbb{N} = \{0, 1, 2, \dots\}$?

Representando tuplas como conjuntos

$$\langle 7, 2 \rangle \stackrel{\text{imp}}{=} \{7, 2\}$$

$$\langle 7, 2 \rangle \stackrel{\text{imp}}{=} \{\{7\}, \{7, 2\}\}$$

$$A \times B \stackrel{\text{imp}}{=} ?$$

$$(TUP1) \quad \langle x, y \rangle = \langle x', y' \rangle \Leftrightarrow x = x' \ \& \ y = y'$$

$$(TUP2) \quad \{ \langle a, b \rangle \mid a \in A, b \in B \} \text{ deve ser um conjunto}$$

Sistemas Hilbert

$$?: \alpha \supset (\beta \supset \alpha)$$

$$?: (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$$

$$?: (\neg \alpha \supset \neg \beta) \supset (\beta \supset \alpha)$$

Como inferir: (Modus Ponens)

Tendo $\alpha \supset \beta$ e α inferir β

$$\vdash A \rightarrow A$$

Gentzen

Natural deduction

Sequent calculus

Natural deduction

$$\frac{A \text{ true} \quad B \text{ true}}{A \& B \text{ true}} \&-I$$

$$\frac{A \& B \text{ true}}{A \text{ true}} \&-E_1$$

$$\frac{A \& B \text{ true}}{B \text{ true}} \&-E_2$$

$$\frac{A \text{ prop} \quad B \text{ prop}}{A \& B \text{ prop}}$$

$$\frac{\frac{A \& B}{B} \quad \frac{A \& B}{A}}{B \& A}$$

comutatividade de (&)

($\in$) vs ($:$), dynamic vs static, ...

$x = \dots$

if (x é um int):

else:

↑
compila

~~int x;~~

~~if (o inteiro x):~~

~~else:~~

↑
não compila

sets
types

escrevemos $a \in A$

escrevemos $a : A$

Set theory ops/construções:
 $\emptyset$, $\{-, -\}$, $\cup$, $\otimes$, $\cap$

Type theory ops/construções:
(outras construções)

Internal ops vs external constructions lec04

2025-03-26

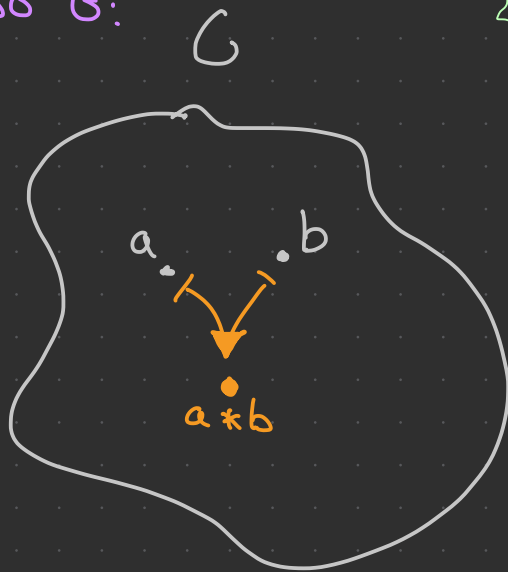
ops internas no grupo G :

$(G ; op, inv, id)$

$op : G \times G \rightarrow G$

$id : G$

$inv : G \rightarrow G$



ops externas ("constructions")

$G \times H, G \oplus H, \text{Aut}(G), \dots$

$A \times B, A \cup B, \wp A, \dots$

Product (binário)

formation

product construction

$$\frac{A \text{ type} \quad B \text{ type}}{A \times B \text{ type}} \quad (x) - F$$

Introduction

pairing

$$\frac{a : A \quad b : B}{\langle a, b \rangle : A \times B} \quad (x) - I$$

$\langle a, b \rangle : A \times B$

Constructor de habitantes de $A \times B$

poderia ter escolhido algo menos

e botar $\langle x, y \rangle \stackrel{\text{sup}}{=} \text{pair } x \ y$

(n-ário?)

$n \geq 3$: fácil $n = 0?$
 $= 1?$

(x) é uma construção
binária entre tipos

Como seria uma construção
nulária? Assim:

$$\frac{}{\mathbb{N} \text{ . Type}} \quad (\mathbb{N}) - F$$

cultural: $\text{pair } a \ b$
 $\text{pair}(a, b)$

Elimination

projections

$$\frac{w : A \times B}{w.l : A} \quad (x)-E_1$$

$$\frac{w : A \times B}{w.r : B} \quad (x)-E_2$$

notações alternativas:

$w.1$
 $w.2$

$w.l$
 $w.r$

$\text{outl } w$
 $\text{outr } w$

$\pi_1 w$
 $\pi_2 w$

Será que $A \times B = A$? $A \times B = \mathbb{N}$?

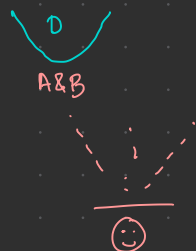
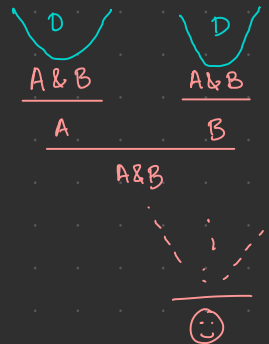
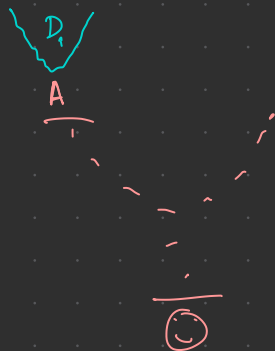
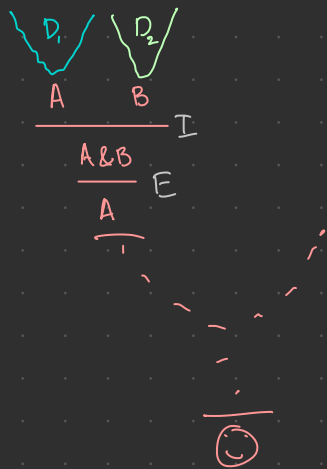
Equations

$$\begin{aligned} \langle a, b \rangle.l &= a \\ \langle a, b \rangle.r &= b \end{aligned} \quad (\beta)$$

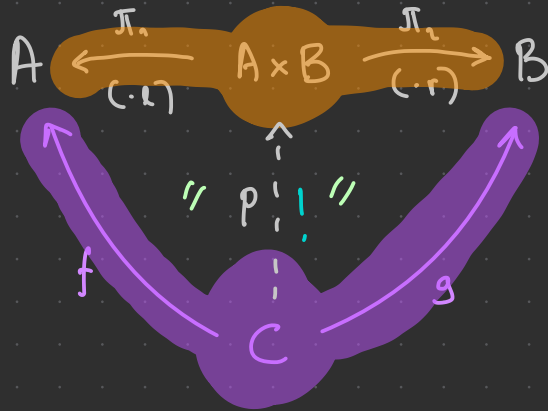
$$(w.l, w.r) = w \quad (\eta)$$

$$\frac{w : A \times B \quad B \text{ type} \quad A \text{ type}}{w.l : A}$$

Proofs compute!



Diagramas comutativos



o diagrama comuta



$$\pi_1 \circ p = f$$

$$\pi_2 \circ p = g$$

$$\langle \pi_1 \circ p, \pi_2 \circ p \rangle$$

$\parallel$

p

$$A \begin{matrix} \xrightarrow{i} \\ \xrightarrow{j} \end{matrix} B \xrightarrow{f} C$$

$$f \circ i = f \circ j$$

(mas não necessariamente $i = j$)

Igualdade

lec05

2025-03-31

~~Definição.~~ Teorema?

Sejam $w, w' : A \times B$.

$$w =_{A \times B} w' \stackrel{\text{def}}{\iff} w.l =_A w'.l \ \& \ w.r =_B w'.r$$

Demons. $(\Rightarrow)$.

$$\frac{\frac{}{w.l = w.l} \text{refl} \quad w = w'}{w.l = w'.l} \text{subs}$$

$$\begin{array}{c} \text{similar} \\ \swarrow \quad \searrow \\ w.r = w'.r \end{array}$$

$(\Leftarrow)$. Calc:

$$\begin{aligned} w &= \langle w.l, w.r \rangle \\ &= \langle w'.l, w'.r \rangle \\ &= w' \end{aligned}$$

$[(\eta)]$

$[\text{hip}; \text{hip}]$

$[(\eta)]$

Categorias

Dados de uma categoria $\mathcal{C}$

$\mathcal{C}_0$ ou $\text{Obj}(\mathcal{C})$: $A, B, C, \dots$

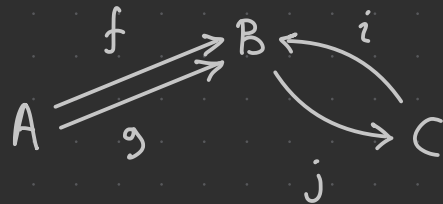
$\mathcal{C}_1$ ou $\text{Arr}(\mathcal{C})$: $f, g, h, \dots$

$\text{src}, \text{tgt} : \text{Arr}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$
source target

Escrevemos como sinônimos:

$A \xrightarrow{f} B$
 $f : A \rightarrow B$
 $f \in \mathcal{C}(A, B)$
 $\text{src}(f) = A \ \& \ \text{tgt}(f) = B$

coleção das setas de A para B



$\text{src}(f) = A$

$\text{tgt}(f) = B$

Operações

Para qualquer objeto A :

$$\text{id}_A : \mathcal{C}(A, A)$$

Para quaisquer objetos A, B, C :

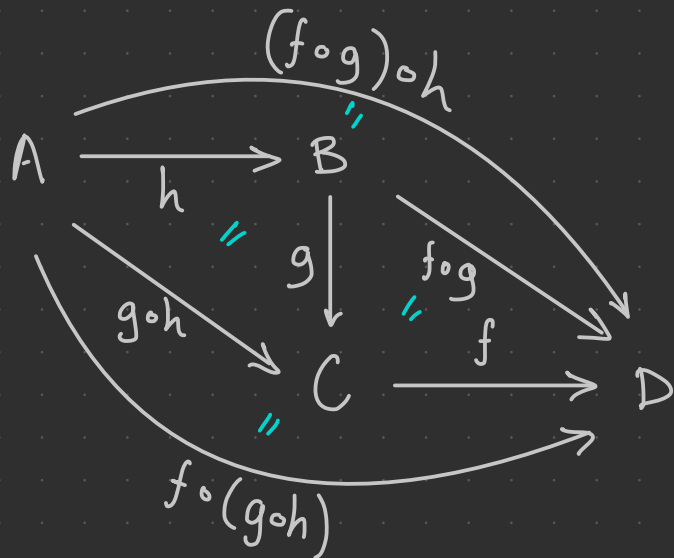
$$\text{comp}_{A,B,C} : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

Sinônimos:

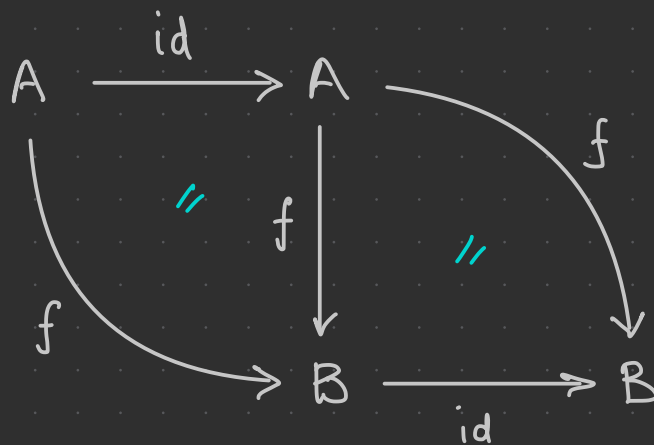
$$\left\{ \begin{array}{ll} \text{id}_A & \text{comp}_{A,B,C}(f, g) \\ 1_A & f \circ g \\ & g ; f \end{array} \right.$$

Leis de categoria

(Ass) $(f \circ g) \circ h = f \circ (g \circ h)$



(Id) $\text{id} \circ f = f$
 $f \circ \text{id} = f$



Exemplos

- Tipos, funções
- Conjuntos, funções
- Proposições, demonstrações
- Grupos, homomorfismos
- Posets, funções monótonas

• 0 : 

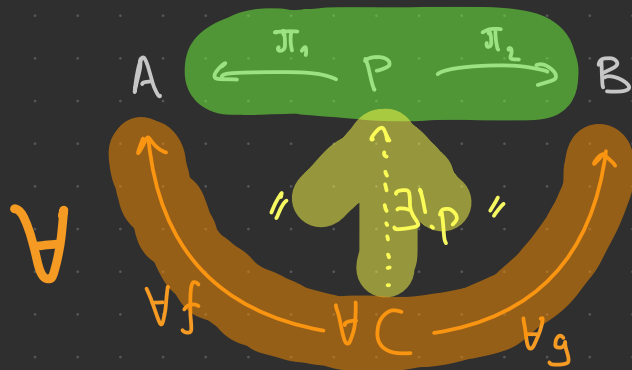
• 1 : 

• 2 : 

• 3 : ?

- Proposições, $(\vdash)$ $a \xrightarrow{f} b$ significa $a \vdash b$
- Inteiros, $(|)$ $a \xrightarrow{f} b$ significa $a | b$
- Inteiros, $(\leq)$ $a \xrightarrow{f} b$ significa $a \leq b$
- Conjuntos, $(\subseteq)$ $A \xrightarrow{f} B$ significa $A \subseteq B$

Universal properties : o produto



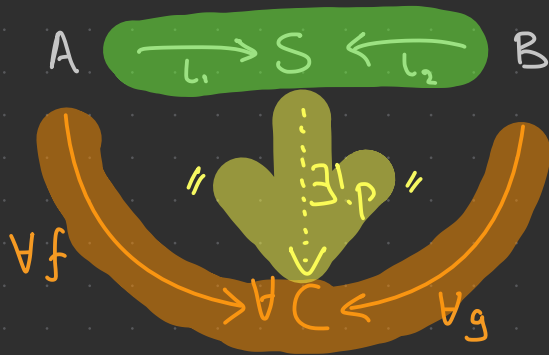
$$? : C \longrightarrow A \times B$$

$$\langle f, g \rangle_c$$

$\xleftarrow{\pi_1} P \xrightarrow{\pi_2}$ é um produto dos A, B
 sse possui tal propriedade universal

Dualidade : coconceito

Coproducto



$\xrightarrow{l_1} S \xleftarrow{l_2}$ é um coproduto dos A, B
sse possui tal propriedade universal

Coproducto: como se manifesta ...

... nos inteiros com (1):

$$(i) \ a | s \ \& \ b | s$$

$$(ii) \ (\forall c) [a | c \ \& \ b | c \Rightarrow s | c]$$

$$(\forall c \text{ mult. com. dos } a, b) [s | c]$$

é o mmc!

...e nas outras cats?

Terminal & inicial

T terminal sse



I cterminal sse
(inicial)



Isomorfismos

$$A \cong B \stackrel{\text{def}}{\iff} (\exists f: A \rightarrow B) [f \text{ iso}]$$

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f'} \end{array} B$$

$$f \text{ iso} \stackrel{\text{def}}{\iff} (\exists f': A \leftarrow B) [f' \circ f = \text{id}_A \ \& \ f \circ f' = \text{id}_B]$$

“Unicidade” (a menos de isomorfismo)

$$(\forall u, v \text{ legais}) [u \cong v]$$

Como entrar vs como sair

$$h : C \rightarrow A \times B$$

$$A \times B \rightarrow C$$

$$h c \stackrel{\text{def}}{=} \left\langle \underbrace{\quad}_{f c}, \underbrace{\quad}_{g c} \right\rangle$$

?

(nada!)

$$f : C \rightarrow A$$

$$f c \stackrel{\text{def}}{=} \underbrace{\quad}_{f c}$$

$$g : C \rightarrow B$$

$$g c \stackrel{\text{def}}{=} \underbrace{\quad}_{g c}$$

Construções de Cats

lec06

2025-04-02

$$\mathbb{C} \mapsto \mathbb{C}^{\text{op}}$$

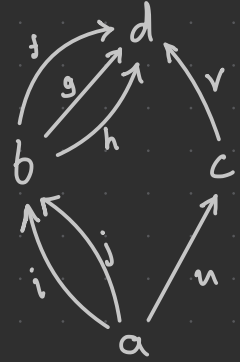
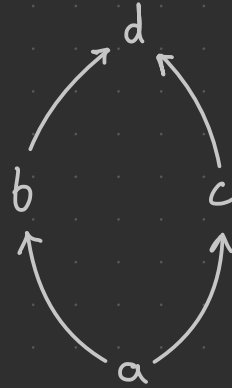
Prosets & Posets vs Cats

(pre order

(partial order

$(P; \leq)$

$(\leq) : P \times P \rightarrow \text{Prop}$



partial order

preorder

Refl : $x \leq x$

Trans : $x \leq y \ \& \ y \leq z \Rightarrow x \leq z$

Antis : $x \leq y \ \& \ y \leq x \Rightarrow x = y$

Sum (coproduct)

formation

$$\frac{A \text{ type} \quad B \text{ type}}{A + B \text{ type}} \quad \text{Soma} \quad (+) - F$$

Introduction

$$\frac{a : A}{\underset{A+B}{l \cdot a} : A + B} \quad \text{L-injection} \quad (+) - I_1$$

$\text{inl } a$

$$\frac{b : B}{\underset{A+B}{r \cdot b} : A + B} \quad \text{R-injection} \quad (+) - I_2$$

$\text{inr } b$

Elimination

$$\frac{s : A + B \quad \overset{\text{contexto}}{\Gamma, a:A \vdash e_1 : C} \quad \overset{\text{case}}{\Gamma, b:B \vdash e_2 : C}}{\Gamma \vdash (\text{case } s \text{ of } | l.a \rightsquigarrow e_1 ; r.b \rightsquigarrow e_2) : C} \text{ (+)-E}$$

$\uparrow$ $\dots a \dots : C$ $\uparrow$ $\dots b \dots : C$

Equations

a e, tendo substituído todas as
ocorrências livres de a por x

$$(\text{case } l.x \text{ of } | l.a \rightsquigarrow e_1 ; r.b \rightsquigarrow e_2) = e_1[a := x]$$

$$(\text{case } r.y \text{ of } | l.a \rightsquigarrow e_1 ; r.b \rightsquigarrow e_2) = e_2[b := y] \text{ (}\beta\text{)}$$

$$(\text{case } s \text{ of } | l.a \rightsquigarrow l.a ; r.b \rightsquigarrow r.b) = s \text{ (}\eta\text{)}$$

Currying

lec07
2025-04-02

$$((\alpha \times \beta) \rightarrow \gamma) \xrightleftharpoons[\text{uncurry}_2]{\text{curry}_2} (\alpha \rightarrow (\beta \rightarrow \gamma)) \quad ?$$

$$\text{curry } f \stackrel{\text{def}}{=} \lambda a : \alpha . \lambda b : \beta . f \langle a, b \rangle$$

$$\frac{\frac{a : \alpha \quad b : \beta}{f : (\alpha \times \beta) \rightarrow \gamma \quad \langle a, b \rangle : \alpha \times \beta} (\times)\text{-I}}{f \langle a, b \rangle : \gamma} (\rightarrow)\text{-E}$$

$$b : \beta$$

$$a : \alpha$$

$$f : \alpha \times \beta \rightarrow \gamma$$

or:

$$\text{curry } f \ a \ b \stackrel{\text{def}}{=} f \langle a, b \rangle$$

$\text{curry}_n, \text{uncurry}_n$: << Similar.>> ? << etc.>> ?

$$\text{curry}_4 f = b \circ c \circ d \stackrel{\text{def}}{=} f \{a, b, c, d\}$$

$$\text{curry}_3 : A \times B \times C \rightarrow D$$

$$\text{curry}_3 : (A \times B) \times C \rightarrow D$$

$$\downarrow \text{curry}_2$$

$$(A \times B) \rightarrow (C \rightarrow D)$$

$$\downarrow \text{curry}_2$$

$$A \rightarrow (B \rightarrow (C \rightarrow D))$$

$$\text{curry}_{n+1} \stackrel{\text{def}}{=} (\text{curry}_2)^{\circ n}$$

$$\text{curry}_3 : A \times (B \times C) \rightarrow D$$

$$\downarrow \text{curry}_2$$

$$A \rightarrow (B \times C \rightarrow D)$$

$$\downarrow ?$$

$$A \rightarrow (B \rightarrow (C \rightarrow D))$$

$$\text{curry}_{n+1} \stackrel{\text{def}}{=} ?$$

Unit

$\frac{\text{nada}}{\text{Unit type}} \text{Unit-F}$
 $\frac{}{\star : \text{Unit}} \text{Unit-I}$
 $\frac{}{(\text{nenhuma})} \text{Unit-E}$

$\mathbb{1} \quad \mathbb{1} \quad \mathbb{1}$

$\text{☺} \quad () \quad \langle \rangle$

?
 B, η?

Bool

$\frac{\text{Bool type}}{\mathbb{B}} \text{Bool-F}$
 $\frac{}{\text{ff} : \text{Bool}} \text{Bool-I}_F$
 $\frac{}{\text{tt} . \text{Bool}} \text{Bool-I}_T$

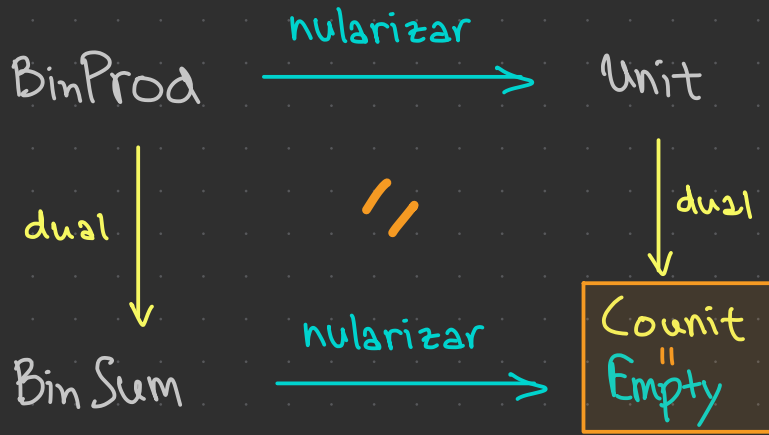
$\frac{b : \text{Bool} \quad y : C \quad n : C}{\text{if } b \text{ then } y \text{ else } n : C} \text{Bool-E}$

(β)

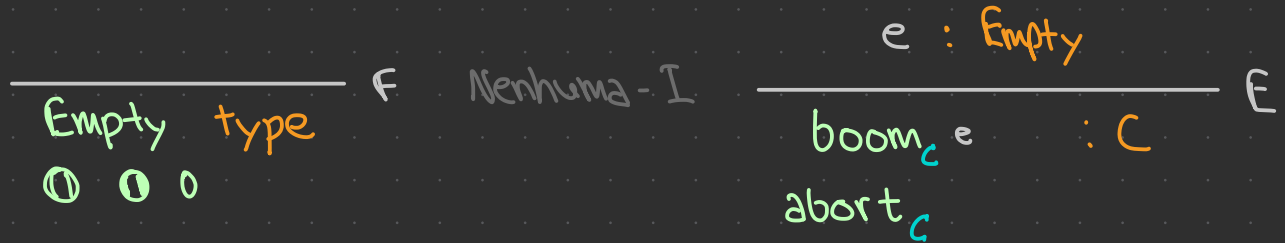
if tt then y else n = y
 if ff then y else n = n

(η)

if b then tt else ff = b



Empty



Quem é o void da C?

β, η ?

Funções de graça

Tendo... será que ganho ..

C
 $\downarrow$
 C'

$$\begin{array}{l} ? \left\{ \begin{array}{l} ((A \times B) \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \\ ((A \times B) \rightarrow (C' \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \end{array} \right. \end{array}$$

B
 $\downarrow$
 B'

$$\begin{array}{l} ? \left\{ \begin{array}{l} ((A \times B) \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \\ ((A \times B') \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \end{array} \right. \end{array}$$

D
 $\downarrow$
 D'

$$\begin{array}{l} ? \left\{ \begin{array}{l} ((A \times B) \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \\ ((A \times B) \rightarrow (C \rightarrow D')) \rightarrow (E + (D' \rightarrow G)) \end{array} \right. \end{array}$$

Quem é o **void** da C?

lec08

2025-04-09

```
int f(void)
    return 42;
```

```
void g(int x)
    return;
```

$$\frac{g : \text{int} \rightarrow \text{void} \quad 42.\text{int}}{g(42) : \text{void}}$$

$f(a_1, \dots, a_n)$ } n-ária

$f_{\perp} \{a_1, \dots, a_n\}$ }

$f_{\perp} a_1_{\perp} \dots_{\perp} a_n$

n-ária descurrificada

n-ária currificada

dá falsa impressão que f recebeu nada

$f()$

deixa claro que f recebeu $()$

$f_{\perp}()$

o único habitante do $\perp$

Function space

$$\frac{A \text{ type} \quad B \text{ type}}{A \rightarrow B \text{ type}} (\rightarrow) - F$$

$$\frac{\Gamma, a:A \vdash b:B}{\Gamma \vdash \lambda a:A. b : A \rightarrow B} (\rightarrow) - I$$

$$\frac{f : A \rightarrow B \quad a:A}{f \text{ } \underline{\quad} a : B} (\rightarrow) - E$$

$\beta?$ $\eta?$

Exemplo:

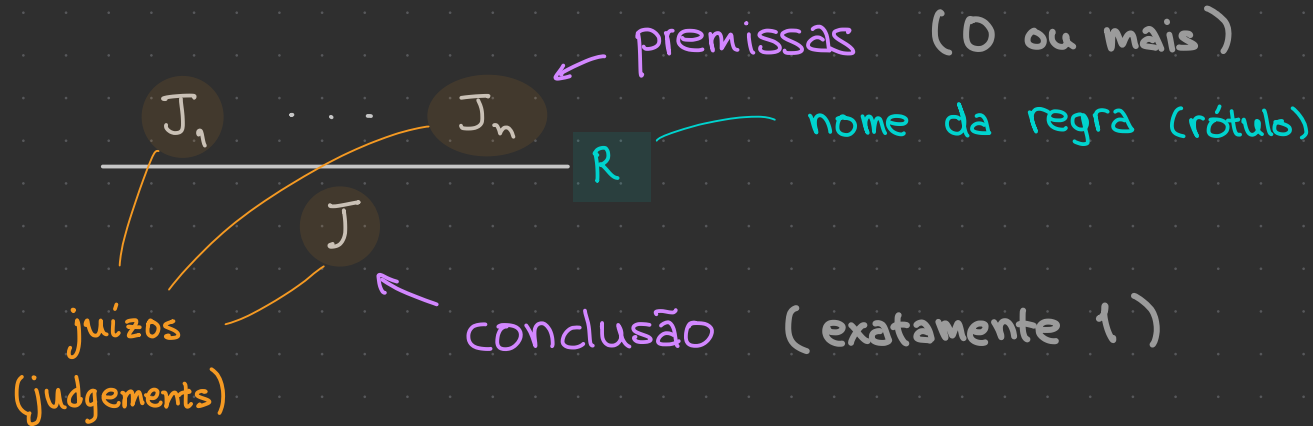
$w : \text{Int}, \quad u : \text{Int}, \quad a : \text{Int}$

$\vdash a + 2 \cdot u : \text{Int}$

$w : \text{Int}, \quad u : \text{Int}$

$\vdash \lambda a:\alpha. a + 2 \cdot u : \text{Int} \rightarrow \text{Int}$

Regras de inferência



Contexts

implícito: $a : A \vdash B$ type

$a : A, b : B, c : C, \dots$

↑
implícito: $\vdash A$ type

↑
implícito: $a : A, b : B \vdash C$ type

$a_1 : A_1, a_2 : A_2(a_1), a_3 : A_3(a_1, a_2), \dots$

Escrevendo: $n : \mathbb{N}, u : \mathbb{R}^n, v : \mathbb{R}^n, m : \mathbb{N}, A : \mathbb{Z}^{n \times m}$

Implícitos: $\vdash \mathbb{N}$ type

$n : \mathbb{N} \vdash \mathbb{R}^n$ type

$n : \mathbb{N}, u : \mathbb{R}^n \vdash \mathbb{R}^n$ type

$n : \mathbb{N}, u : \mathbb{R}^n, v : \mathbb{R}^n \vdash \mathbb{N}$ type

$n : \mathbb{N}, u : \mathbb{R}^n, v : \mathbb{R}^n, m : \mathbb{N} \vdash \mathbb{Z}^{n \times m}$ type

Dependent types / Families

" $A : I \rightarrow \text{Type}$ "
indexing

$i : I \vdash B(i) \text{ type}$

$i : I \vdash B(i) \text{ type}$

B é uma família de tipos
(indexada por $i : I$)

B é um tipo dependente
(do $i : I$)

Exemplo:

" $\mathbb{R}^\square : \mathbb{N} \rightarrow \text{Type}$ "

$n : \mathbb{N} \vdash \mathbb{R}^n \text{ type}$

Exemplo:

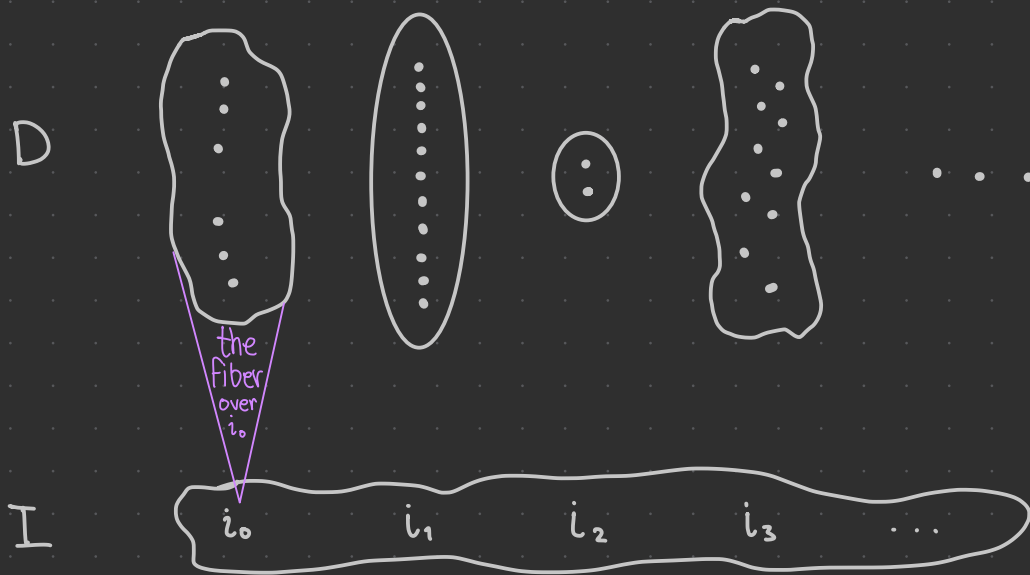
" $A : \mathbb{B} \rightarrow \text{Type}$ "

$A(\text{ff}) \stackrel{\text{def}}{=} \mathbb{N}$

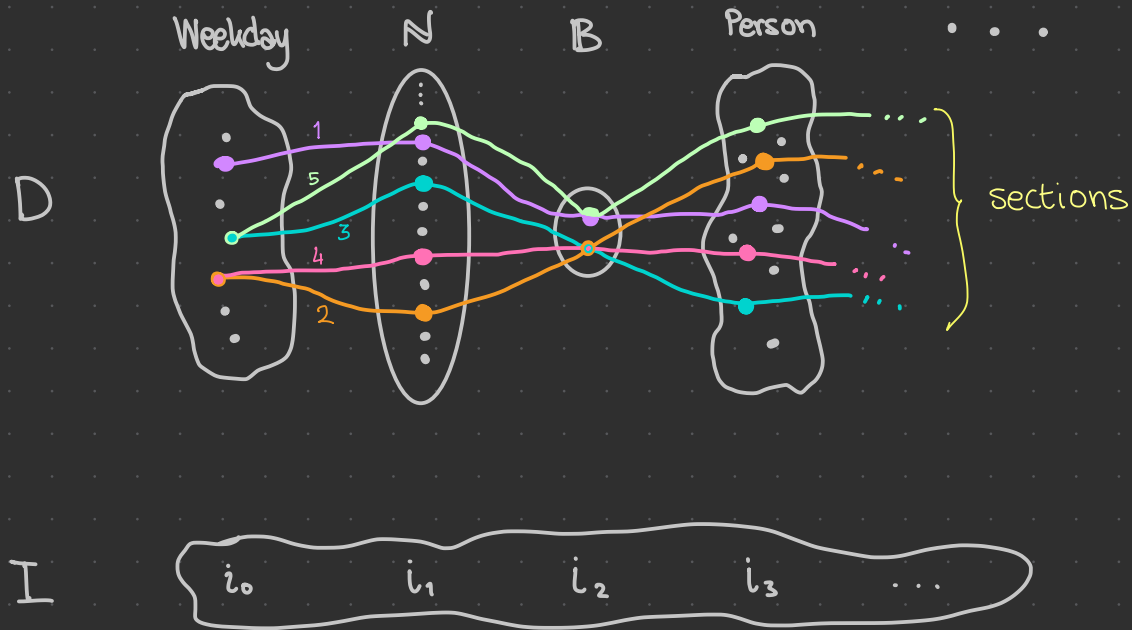
$A(\text{tt}) \stackrel{\text{def}}{=} \mathbb{N} \times (\mathbb{N} \rightarrow \mathbb{N})$

$b : \mathbb{B} \vdash A(b) \text{ type}$

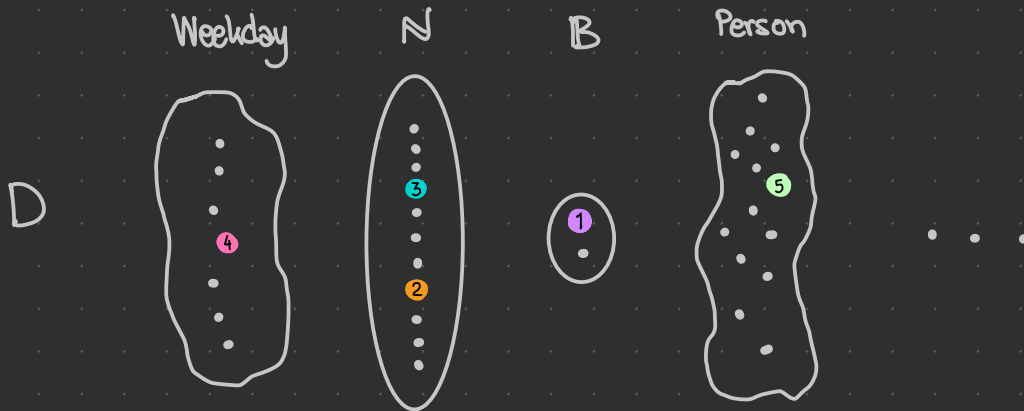
$i : I \vdash D(i) \text{ type}$

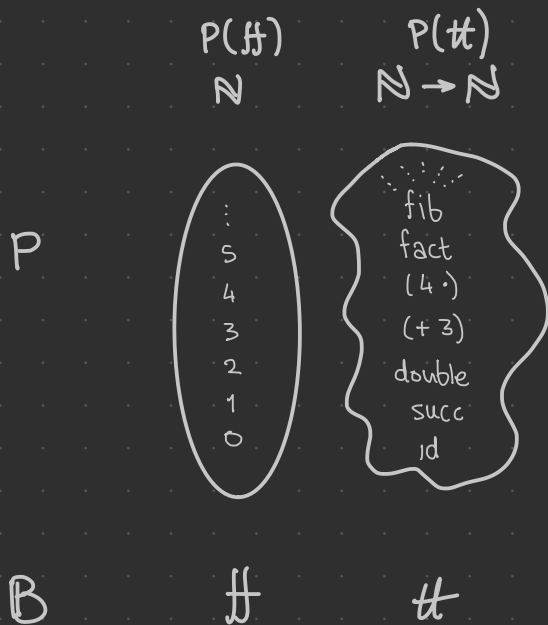


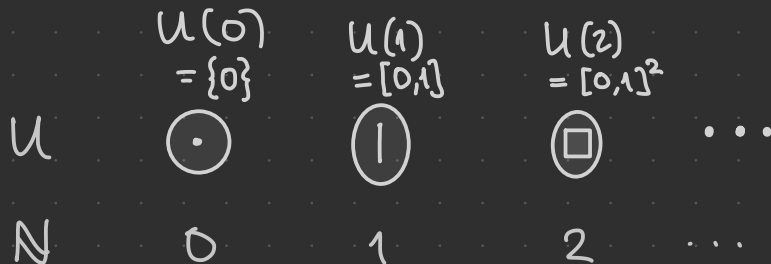
5 habitantes do TD

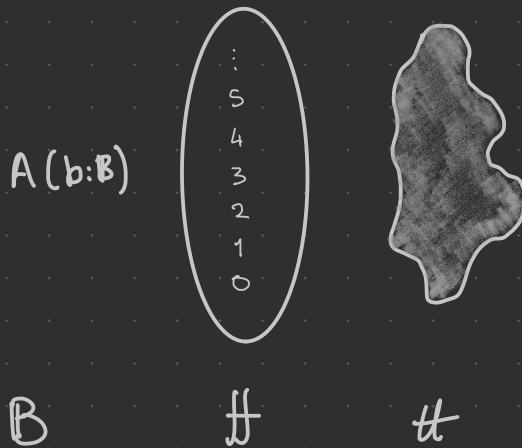


5 habitantes do ΣD



$$b : \mathbb{B} \vdash P(b) \text{ type}$$


$$n : \mathbb{N} \vdash U(n) \text{ type}$$


$$b : \mathbb{B} \vdash A(b) \text{ type}$$


Π (Dependent product)

A é um dependent type

$$\boxed{i : I \vdash A(i) \text{ type}}$$

(Π) -F

$$\vdash \prod_{i:I} A(i) \text{ type}$$

$$\Pi(i:I).A(i)$$

$$(i:I) \rightarrow A(i)$$

$$? : (b:B) \rightarrow A(b)$$

I?

E?

β, η ?

Σ (Dependent sum)

$$\frac{i : I \vdash A(i) \text{ type}}{(\Sigma)\text{-F}}$$

$$\vdash \sum_{i:I} A(i) \text{ type}$$

$$\sum (i:I). A(i)$$

$$(i:I) \times A(i)$$

$$? : (b:B) \times A(b)$$

I? E? β, η ?

Isomorfismos

lec09

2025-04-14

$$\alpha \cong \beta \stackrel{\text{def}}{\iff} (\exists f: \alpha \rightarrow \beta) [f \text{ iso}]$$

$$f: \alpha \rightarrow \beta \text{ iso} \stackrel{\text{def}}{\iff} (\exists f': \alpha \leftarrow \beta) [\underbrace{f' \text{ inversa de } f}_{f' \circ f = \text{id}_\alpha \ \& \ f \circ f' = \text{id}_\beta}]$$

invertível

$$\alpha \xrightarrow{\cong} \beta$$

$$f: \alpha \cong \beta$$

Θ . $(\cong)$ é uma relação de equivalência:

$(\cong)$ - refl

$(\cong)$ - trans

$(\cong)$ - sym

$$f^{-1} \stackrel{\text{def}}{=} a \text{ inversa de } f$$

Aritmética de tipos

$$(\alpha + \beta) + \gamma \cong \alpha + (\beta + \gamma)$$

$$0 + \alpha \cong \alpha$$

$$\alpha + \beta \cong \beta + \alpha$$

$$(\alpha \times \beta) \times \gamma \cong \alpha \times (\beta \times \gamma)$$

$$1 \times \alpha \cong \alpha$$

$$\alpha \times \beta \cong \beta \times \alpha$$

Distributividades:

$$\delta \times (\alpha + \beta) \cong (\delta \times \alpha) + (\delta \times \beta)$$

$$\delta \times 0 \cong 0$$

$$\delta \times \sum_{i \in I} \alpha_i \cong \sum_{i \in I} (\delta \times \alpha_i)$$

Exemplo:

$$f : \delta \times (\alpha + \beta) \cong (\delta \times \alpha) + (\delta \times \beta)$$

$$f \langle d, l \cdot a \rangle \stackrel{\text{def}}{=} l \cdot \langle d, a \rangle$$

$$f \langle d, r \cdot b \rangle \stackrel{\text{def}}{=} r \cdot \langle d, b \rangle$$

Sem pattern-matching:

$$f \ w \stackrel{\text{def}}{=} \text{case } w \text{ of } \left[\begin{array}{l} l \ a \rightsquigarrow l \cdot \langle w \cdot l, a \rangle \\ r \cdot b \rightsquigarrow r \cdot \langle w \cdot l, b \rangle \end{array} \right]$$

Conclusão: ❤ pattern-matching

Exponenciação

$$\alpha^{\beta+\gamma} \cong \alpha^{\beta} \times \alpha^{\gamma}$$

$$\alpha^{\sum_{i \in I} \beta_i} \cong \prod_{i \in I} \alpha^{\beta_i}$$

versão
I-ária

versão
nulária

$$\alpha^0 \cong 1$$

$$\alpha^{\beta \times \gamma} \cong (\alpha^{\beta})^{\gamma}$$

currificação

Conclusão: Aprendi currificação
no ensino médio !?



$$\alpha^{?(\beta+\gamma)} \cong (\alpha^{? \beta}) \times (\alpha^{? \gamma})$$

$$\alpha^{?(\beta \times \gamma)} \cong (\alpha^{? \beta})^{? \gamma}$$

$$(?) \equiv (\leftarrow)$$

$$\alpha^{\beta} \equiv_{\text{def}}$$

$$\beta \rightarrow \alpha$$

ou

$$\alpha \leftarrow \beta$$

ou

$$\alpha \xleftarrow{\beta}$$

Cardinalidades

$$|\alpha \times \beta| = |\alpha| \cdot |\beta|$$

$$|\alpha + \beta| = |\alpha| + |\beta|$$

$$|\alpha \leftarrow \beta| = |\alpha|^{|\beta|}$$

$$2 \stackrel{\text{def}}{=} 1 + 1 \cong \mathbb{B} \quad \alpha^0 \cong 1 \quad (\alpha + \beta)^2 \cong ?$$

$$3 \cong 2 + 1 \quad \alpha^1 \cong \alpha$$

$$\alpha^2 \cong \alpha \times \alpha$$

$$\vdots \quad \vdots$$

$\aleph$

$$A + B$$

$$A \times B$$

$$A \rightarrow B$$

Caso binário
 $I := 2, D(0) := A, D(1) := B$

 Σ

família invariável
 $(I := A, D(0) := B)$

Caso binário
 $I := 2, D(0) := A, D(1) := B$

 Π

família invariável
 $(I := A, D(0) := B)$

família invariável: $I := A; D(a) := B$

$$\sum_{(a:A)} D(a) \equiv (a:A) \times B \equiv A \times B$$

$$\prod_{(a:A)} D(a) \equiv (a:A) \rightarrow B \equiv A \rightarrow B$$

caso binário: $I := 2; D(0) := A; D(1) := B$

$$\sum_{(i:2)} D(i) \cong D(0) + D(1) \equiv A + B$$

$$\prod_{(i:2)} D(i) \cong D(0) \times D(1) \equiv A \times B$$

Propositions as Types (Curry-Howard)

Certos tipos podem ser vistos como proposições e seus habitantes como evidências - demonstrações.

(as construções dos)

Logic	Types
$\perp$	$\mathbf{0}$
$\top$	$\mathbf{1}$
$P \wedge Q$	$P \times Q$
$P \Rightarrow Q$	$P \rightarrow Q$
$P \vee Q$	$P + Q$
$\forall x \in A. P(x)$	$\prod (x:A). P(x)$
$\exists x \in A. P(x)$	$\sum (x:A). P(x)$
$u \stackrel{?}{=}_A v$	$\text{Id}_A(u, v)$

$$\neg P \equiv P \Rightarrow \perp$$

$$P \Leftrightarrow Q \equiv \dots$$

truncamento

$$\| P \| \stackrel{\text{def}}{=} \neg \neg P$$

as identificações dos $u, v : A$

decsão

disjunção

construção

existencial

Previously ... Funções de graça

lec10

2025-04-16

Tendo... Será que ganho ..

C
↓
C'

$$\begin{aligned} & \left(((A \times B) \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \right) \\ ? \quad & \left(((A \times B) \rightarrow (C' \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \right) \end{aligned}$$

B
↓
B'

$$\begin{aligned} & \left(((A \times B) \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \right) \\ ? \quad & \left(((A \times B') \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \right) \end{aligned}$$

D
↓
D'

$$\begin{aligned} & \left(((A \times B) \rightarrow (C \rightarrow D)) \rightarrow (E + (D \rightarrow G)) \right) \\ ? \quad & \left(((A \times B) \rightarrow (C \rightarrow D')) \rightarrow (E + (D' \rightarrow G)) \right) \end{aligned}$$

$$\begin{array}{ccc}
 A & A \times B & \\
 f \downarrow & \downarrow? & (a, b) \mapsto (f a, b) \\
 A' & A' \times B &
 \end{array}$$

$$\begin{array}{ccc}
 B & A \times B & \\
 \downarrow g & \downarrow? & \text{Similar} \\
 B' & A \times B' &
 \end{array}$$

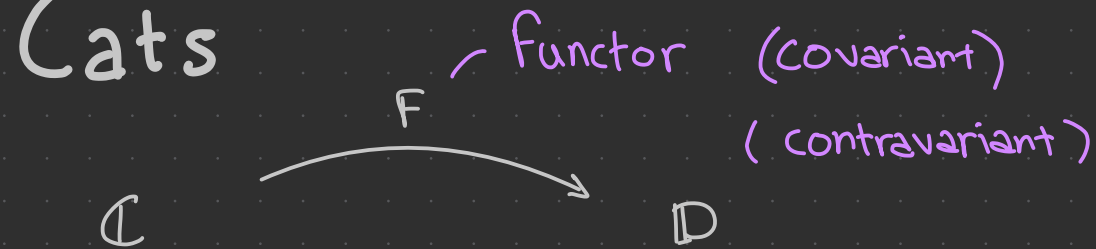
$$\begin{array}{ccc}
 A & A + B & (l \cdot a) \mapsto l \cdot f a \\
 f \downarrow & \downarrow? & (r \cdot b) \mapsto r \cdot b \\
 A' & A' + B &
 \end{array}$$

$$\begin{array}{ccc}
 B & A \times B & \\
 \downarrow g & \downarrow? & \text{Similar} \\
 B' & A \times B' &
 \end{array}$$

$$\begin{array}{ccc}
 A & A \rightarrow B & \text{pre comp.} \\
 f \downarrow & ? \uparrow \downarrow ? & (\circ f) \\
 A' & A' \rightarrow B & u' \mapsto \lambda a. u'(f a)
 \end{array}$$

$$\begin{array}{ccc}
 B & A \xrightarrow{u} B & \text{post-comp} \\
 g \downarrow & \downarrow? & (g \circ) \\
 B' & A \rightarrow B' & u \mapsto \lambda a. \underbrace{g(u a)}_{(g \circ u) a}
 \end{array}$$

Cats



$$\text{src} \circ F = F \circ \text{src}$$

$$\text{tgt} \circ F = F \circ \text{tgt}$$

$$F(g \circ f) = Fg \circ Ff$$

$$F(\text{id}_A) = \text{id}_{FA}$$

comp. respeita functorialidade

$$F_0 : \mathbb{C}_0 \rightarrow \mathbb{D}_0$$

$$F_1 : \mathbb{C}_1 \rightarrow \mathbb{D}_1$$

hw

class Functor $F : Ty \rightarrow Ty$

fmap : $(A \xrightarrow{f} B) \xrightarrow{F} (F A \xrightarrow{F f} F B)$

(+ leis)

$Int \xrightarrow{\quad} List\ Int$

$(Int \rightarrow List\ Int) \xrightarrow{List} List\ (Int \rightarrow List\ Int)$

(o Functor List
sendo aplicado em objetos

instance Functor List : $Ty \rightarrow Ty$

fmap : $(A \rightarrow B) \rightarrow (List\ A \rightarrow List\ B)$

List

List₀ : $Type \rightarrow Type$

List₁ : $(A \rightarrow B) \rightarrow (List_0\ A \rightarrow List_0\ B)$

fmap

$$D \xrightarrow{(A \times -)} A \times D$$

$$\begin{array}{ccc} D & & A \times D \\ f \downarrow & \xrightarrow{(A \times -)} & \downarrow A \times f \\ D' & & A \times D' \end{array}$$

$(\times)$: covariante

$(+)$: covariante

$(- \rightarrow B)$: contravariante

$(A \rightarrow -)$: covariante

Σ, Π

$$\frac{\Gamma \vdash I \text{ type} \quad \Gamma, i:I \vdash A_i \text{ type}}{\Sigma\text{-F}}$$

$$\Gamma \vdash (i:I) \times A_i \text{ type}$$

on $\sum_{i:I} A_i$ on $\sum i:I. A_i$

$$\frac{\Gamma \vdash I \text{ type} \quad \Gamma, i:I \vdash A_i \text{ type}}{\Pi\text{-F}}$$

$$\Gamma \vdash (i:I) \rightarrow A_i \text{ type}$$

on $\prod_{i:I} A_i$ on $\prod i:I. A_i$

$$\Gamma \vdash i : I$$

$$\Gamma \vdash a_i : A_i$$

$$\frac{}{\Gamma \vdash \langle i, a_i \rangle : (i : I) \times A_i} \Sigma-I$$

$$\Gamma, i : I \vdash a_i : A_i$$

$$\frac{}{\Gamma \vdash \lambda i : I. a_i : (i : I) \rightarrow A_i} \Pi-I$$

$$\frac{\Gamma \vdash w : (i : I) \times A_i}{\Gamma \vdash w.l : I} \Sigma-E$$

$$\Gamma \vdash w.r : A_{w.l}$$

$$\Gamma \vdash w.r : A_{w.l}$$

$$\Gamma \vdash f : (i : I) \rightarrow A_i$$

$$\Gamma \vdash j : I$$

$$\frac{}{\Gamma \vdash f_{_} j : A_j} \Pi-E$$

Universos

Universe : um tipo cujos elementos são tipos.

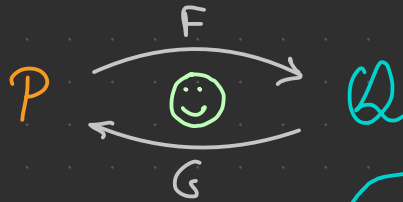
~~Type : Type~~ / Girard

Type₀ : Type₁ : Type₂ : ...

quão longa?

Galois connections

lec11
2025-04-23



como
comparar
os p, q ?



Quais são as setas?

Funções monótonas:

$$x \leq_P y \Rightarrow f x \leq_Q f y$$

← Galois (connection)

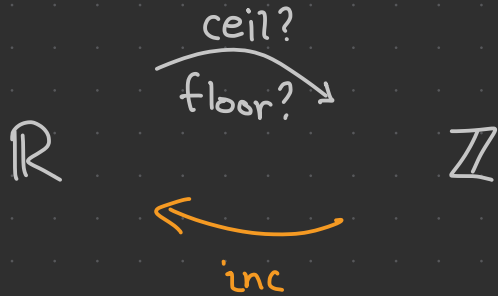
F  G

↑ lower
ou
left adjoint upper
ou
right

def
 $\iff$

$$F p \leq_Q q \iff p \leq_P G q$$

Θ. "dado um dos dois, o outro é determinado"



inc ☺ ?

? ☺ inc

$\mathbb{N}$

$$\frac{}{\mathbb{N} \text{ type}} \quad \mathbb{N}-F$$

$$\frac{}{0 : \mathbb{N}} \quad \mathbb{N}-I_z$$

$$\frac{n : \mathbb{N}}{\text{succ}(n) : \mathbb{N}} \quad \mathbb{N}-I_s$$

$$\Gamma \vdash P : \mathbb{N} \rightarrow \text{Type}$$

$$\Gamma \vdash b : P(0)$$

$$\Gamma, k : \mathbb{N}, h : P(k) \vdash i(k, h) : P(\text{succ}(k))$$

$$\Gamma \vdash n : \mathbb{N}$$

$$\frac{}{\Gamma \vdash \text{ind}_{\mathbb{N}}(P, b, i, n) : P(n)} \quad \mathbb{N}-E$$



Equações:

$$\text{ind}_{\mathbb{N}}(P, b, i, 0) = b \quad : P(0)$$

$$\text{ind}_{\mathbb{N}}(P, b, i, \text{succ}(k)) = i(k, \text{ind}_{\mathbb{N}}(P, b, i, k)) : P(\text{succ}(k))$$

MLTT

└ Martin Löf Type Theory

lec12

2025-04-28

$\frac{\mathcal{H}_1 \dots \mathcal{H}_n}{\quad} \text{label}$
C

juizos

$\frac{\Gamma \vdash a : A \quad \Gamma \vdash f : A \rightarrow B}{\Gamma \vdash f a : B}$

(i) A é um tipo bem formada dentro do contexto Γ .

$$\Gamma \vdash A \text{ type}$$

judgementally equal

(ii) A, B são tipos **juizamente iguais** dentro do contexto Γ .

$$\Gamma \vdash A \doteq B \text{ type}$$

(iii) a é um elemento do tipo A no ctx Γ

$$\Gamma \vdash a : A$$

(iv) a, b são elementos **mente iguais** do tipo A no Γ

$$\Gamma \vdash a \doteq b : A$$

$$\Gamma \vdash J$$

Contexto:

$x_1 : A_1, x_2 : A_2(x_1), x_3 : A_3(x_1, x_2), \dots, x_n : A_n(x_1, \dots, x_{n-1})$

Para cada $1 \leq k \leq n$:

$x_1 : A_1, \dots, x_{k-1} : A_{k-1}(x_1, \dots, x_{k-2}) \vdash A_k(x_1, \dots, x_{k-1}) \text{ type}$

Nomes únicos!

Processo recursivo de verificação de ctx (de L para R).

Famílias de tipos

$\Gamma, x:A \vdash B(x)$ type

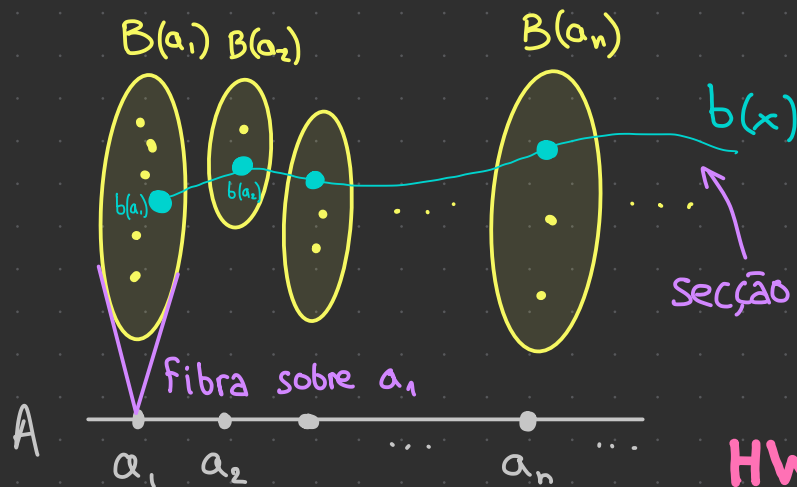
$A \rightarrow \text{Type}$

B é uma fam de tipos indexada por $x:A$

Exemplo:

$n : \mathbb{N} \vdash \mathbb{R}^n$ type

$\Gamma, x:A \vdash b(x) : B(x)$



HW

Regras

(i) Sobre info implícitas de formação de: ctx, tipos, elems

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash A \text{ type}} \text{idx-type}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash A \text{ type}} \text{elem-type}$$

$$\frac{\Gamma \vdash A \doteq B \text{ type}}{\begin{array}{l} \Gamma \vdash A \text{ type} \\ \Gamma \vdash B \text{ type} \end{array}} \begin{array}{l} \text{jeq-L} \\ \text{typejeq-type} \\ \text{-L} \\ \text{-R} \end{array}$$

$$\frac{\Gamma \vdash a \doteq b : A}{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma \vdash b : A \end{array}} \begin{array}{l} \text{jeq-L} \\ \text{elemjeq-elem} \\ \text{-L} \\ \text{-R} \end{array}$$

HW

$$\frac{\Gamma, x:A, \Delta \vdash J}{\Gamma \vdash A \text{ type}} \text{ctx-type}$$

$$\frac{\Gamma \vdash a \doteq b : A}{\frac{\Gamma \vdash a : A}{\Gamma \vdash A \text{ type}}}$$

derivável:

$$\frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash A \text{ type}} \text{elemjeq-type}$$

(ii) Que estabelecem que ($\doteq$) é uma rel de equivalência

typejeq-

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \doteq A \text{ type}} \text{ jeq-refl}$$

-refl

$$\frac{\begin{array}{l} \Gamma \vdash A \doteq B \text{ type} \\ \Gamma \vdash B \doteq C \text{ type} \end{array}}{\Gamma \vdash A \doteq C \text{ type}} \text{ jeq-trans}$$

-trans

$$\frac{\Gamma \vdash A \doteq B \text{ type}}{\Gamma \vdash B \doteq A \text{ type}} \text{ jeq-sym}$$

-sym

elemjeq-

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \doteq a : A} \text{ jeq-refl}$$

$$\frac{\begin{array}{l} \Gamma \vdash a \doteq b : A \\ \Gamma \vdash b \doteq c : A \end{array}}{\Gamma \vdash a \doteq c : A} \text{ jeq-trans}$$

$$\frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash b \doteq a : A} \text{ jeq-sym}$$

(iii) conversões

conv-var

conversão de variável

scheme:
(4-in-1)

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma, x:A, \Delta \vdash J}{\Gamma, x:A', \Delta \vdash J} \text{conv}(x:A \rightarrow A')\text{-var}$$

Deriváveis: (HW)

conv-lem

conversão de elemento

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma \vdash a:A}{\Gamma \vdash a:A'} \text{conv}(a:A \rightarrow A')\text{-elem}$$

conv-elemjeq

conversão de elemjeq

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A'} \text{conv}(a \doteq b : A \rightarrow A')\text{-elemjeq}$$

(iv) Substituições

sub-elem

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash J}{\Gamma, \Delta[x := a] \vdash J[x := a]} \text{sub}(x := a)_{\text{-elem}}$$

$$e[x := a] \stackrel{\text{def}}{=}$$

Também usado, mas vamos evitar: $e[a/x]$

e substituindo todas as
ocorrências de x por a
tomando os devidos cuidados

$$\frac{\begin{array}{l} \Gamma \vdash a : A \\ \Gamma, x : A, y_1 : B_1, \dots, y_n : B_n \vdash C \text{ type} \end{array}}{\Gamma, y_1 : B_1[x := a], \dots, y_n : B_n[x := a] \vdash C[x := a] \text{ type}}$$

sub-typeeq

$$\frac{\Gamma \vdash a \doteq a' : A \quad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[x := a] \vdash B[x := a] \doteq B[x := a'] \text{ type}} \text{sub}(x := a, a')_{\text{-typeeq}}$$

sub-elemjeq

$$\frac{\Gamma \vdash a \doteq a' : A \quad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[x := a] \vdash b[x := a] \doteq b[x := a'] : B[x := a]} \text{sub}(x := a, a')_{\text{-elemjeq}}$$

sub-typejeq & sub-elemjeq as congruence rules

cong-jeq-type (sub-typejeq)

$$\frac{a \doteq a' : A \quad B(x:A) \text{ type}}{B(a) \doteq B(a') \text{ type}} \text{cong-jeq}(B[x:=a,a'])_{\text{-type}}$$

cong-jeq-elem (sub-elemjeq)

$$\frac{a \doteq a' : A \quad b(x:A) : B(x:A)}{b(a) \doteq b(a') : B(a)} \text{cong-jeq}(b[x:=a,a'])_{\text{-elem}}$$

(v) Weakening

lec13

2025-04-30

weak

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, \Delta \vdash J}{\Gamma, x:A, \Delta \vdash J} \quad \begin{array}{l} \text{weak}(\Gamma; x:A) * \\ \text{wkg} \end{array} \quad x \text{ fresco} \\ \text{(em } \Gamma, \Delta)$$

CASO ESPECIAL:

kfam

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x:A \vdash B \text{ type}} \quad \text{kfam}(x:A, B) *$$

(formação de família constante (trivialmente dependente)

...finalmente:

(vi) generalized elem, identity, $\vdash$ -refl, "axiom", ...

gen

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x:A \vdash x:A} \begin{array}{l} \text{gen}(x:A) \\ \delta(x:A) \end{array} *$$

Derivações

$$\frac{\mathcal{H}_1 \quad \dots \quad \mathcal{H}_n}{\quad} \quad \subset$$

rename

$$\frac{\Gamma, x:A, \Delta \vdash J}{\Gamma, x':A, \Delta[x:=x'] \vdash J[x:=x']} \text{ rename}(x, x') * x' \text{ fresco (no ctx)}$$

HW

$$\begin{array}{c} \text{?} \\ \swarrow \quad \searrow \\ \Gamma \vdash A \text{ type} \end{array}$$

gen(x':A)

$$\Gamma, x'.A \vdash x':A$$

$$\begin{array}{c} \text{?} \\ \swarrow \quad \searrow \\ \Gamma \vdash A \text{ type} \end{array}$$

$$\Gamma, x:A, \Delta \vdash J$$

* w(Γ; x'.A)

$$\Gamma, x':A, x:A, \Delta \vdash J$$

sub(x:=x')
-elem

$$\Gamma, x':A, \Delta[x:=x'] \vdash J[x:=x']$$

swap

Qual é o papel disso?

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x:A, y:B, \Delta \vdash J}{\Gamma, y:B, x:A, \Delta \vdash J} \text{swap}(x,y)$$

$$\frac{\Gamma \vdash B \text{ type} \quad \frac{\Gamma \vdash B \text{ type} \quad \frac{\Gamma, x:A, y:B, \Delta \vdash J}{\Gamma, x:A, y':B, \Delta[y:=y'] \vdash J[y:=y']} \text{rename}(y,y') *}{\Gamma, y:B, x:A, y':B, \Delta[y:=y'] \vdash J[y:=y']} \text{weak}(\Gamma; y:B)}{\Gamma, y:B, x:A, \Delta \vdash J} \text{sub}_{\text{elem}}(y' := y)$$

$$\Gamma, y:B, x:A, \boxed{\Delta} \vdash \boxed{J}$$

* *

$$\boxed{\Delta[y:=y'][y' := y]} \quad \boxed{J[y:=y'][y' := y]}$$

versões $\leftarrow$ e $-R$

conv-var $\leftarrow$

$$\frac{\Gamma \vdash A' \doteq A \text{ type} \quad \Gamma, x:A, \Delta \vdash J}{\Gamma, x:A', \Delta \vdash J} \begin{array}{l} \text{conv}(x:A' \leftarrow A) \\ \text{-var} \\ \text{conv}(x:A \rightarrow A') \end{array}$$

conv-elem $\leftarrow$

$$\begin{array}{l} \text{conv}(a : A' \leftarrow A) \\ \text{conv}(a : A \rightarrow A') \end{array}$$

As: typejeq - type - R são deriváveis.
elemjeq - elem - R

conv-elemjeq $\leftarrow$

$$\begin{array}{l} \text{conv}(a \doteq b : A' \leftarrow A) \\ \text{conv}(a \doteq b : A \rightarrow A') \end{array}$$

HW

conv-elem

conversão de elemento

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma \vdash a : A}{\Gamma \vdash a : A'} \text{ conv } (a : A \rightsquigarrow A') \text{ -elem}$$

conv-elemjeq

conversão de elemjeq

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A'} \text{ conv } (a \doteq b : A \rightsquigarrow A') \text{ -elemjeq}$$

ctx-last

$$\frac{\Gamma, x : A \vdash J}{\Gamma \vdash A \text{ type}} \text{ ctx-last}$$

ctx

type-validity of context

$$\frac{\Gamma, x : A, \Delta \vdash J}{\Gamma \vdash A \text{ type}} \text{ ctx}$$

conv-elem

lec14

2025-05-05

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma \vdash a : A}{\Gamma \vdash a : A'} \text{conv (} a : A \rightsquigarrow A' \text{)}_{\text{-elem}}$$

$$\frac{\Gamma \vdash A \doteq A' \text{ type}}{\Gamma \vdash A' \text{ type}} \text{jeq-R}$$

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma, x : A' \vdash x : A'}{\Gamma, x : A' \vdash x : A'} \text{gen (} x : A' \text{)}$$

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma, x : A' \vdash x : A'}{\Gamma, x : A' \vdash x : A'} \text{conv}^{\leftarrow} (x : A' \rightsquigarrow A)_{\text{-var}}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A \vdash x : A'}{\Gamma \vdash a : A'} \text{sub (} x := a \text{)}_{\text{-elem}}$$

$$\Gamma \vdash a : A'$$

$$(x : A')[x := a]$$

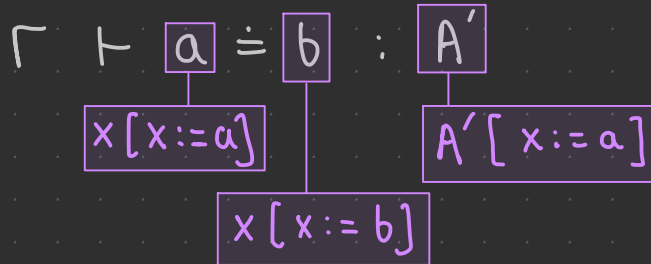
conv-elemjeq

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A'} \text{ conv (} a \doteq b : A \rightsquigarrow A' \text{) -elemjeq}$$

$$\frac{\Gamma \vdash A \doteq A' \text{ type}}{\Gamma \vdash A' \text{ type}} \text{ jeq-R}$$

$$\frac{\Gamma \vdash A \doteq A' \text{ type} \quad \frac{\Gamma, x:A' \vdash x:A'}{\Gamma, x:A \vdash x:A'} \text{ gen (} x:A' \text{) conv}^{\leftarrow} \text{(} x:A' \rightsquigarrow A \text{) -var}}{\Gamma, x:A \vdash x:A'}$$

$$\frac{\Gamma \vdash a \doteq b : A}{\Gamma \vdash a \doteq b : A} \text{ sub-ejeq (} x:=a, b \text{)}$$



ctx-last

$$\frac{\Gamma, x:A \vdash J}{\Gamma \vdash A \text{ type}} \text{ ctx-last}$$

$$\frac{\Gamma, x:A \vdash B \text{ type}}{\Gamma \vdash A \text{ type}} \text{ idx-type}$$

$$\frac{\Gamma, x:A \vdash B \doteq B' \text{ type}}{\Gamma, x:A \vdash B \text{ type}} \text{ jeq-L}$$
$$\frac{}{\Gamma \vdash A \text{ type}} \text{ idx-type}$$

$$\frac{\Gamma, x:A \vdash b:B}{\Gamma, x:A \vdash B \text{ type}} \text{ elem-type}$$
$$\frac{}{\Gamma \vdash A \text{ type}} \text{ idx-type}$$

$$\frac{\Gamma, x:A \vdash b \doteq b' : B}{\Gamma, x:A \vdash B \text{ type}} \text{ ejeq-type}$$
$$\frac{}{\Gamma \vdash A \text{ type}} \text{ idx-type}$$

ctx

$$\frac{\Gamma, x:A, \Delta \vdash J}{\Gamma \vdash A \text{ type}} \text{ ctx}$$

Por indução (recursão?) no Δ :

BASE ($\Delta =: []$):

(ctx-last)

PASSO INDUTIVO ($\Delta =: \Delta', z:Z$):

$$\frac{\Gamma, x:A, \Delta', z:Z \vdash J}{\Gamma, x:A, \Delta' \vdash Z \text{ type}} \text{ ctx-last}$$
$$\frac{\Gamma, x:A, \Delta' \vdash Z \text{ type}}{\Gamma \vdash A \text{ type}} \text{ (I.H.)}$$

Π : produto (de família de tipos)
ou tipo de funções dependentes

Π -formação

$$\frac{\Gamma, x:A \vdash B(x) \text{ type}}{\Gamma \vdash \Pi_{(x:A)} B(x) \text{ type}} \text{ } \Pi\text{-F}$$

$(x:A) \rightarrow B(x)$

Cong:

$$\frac{\begin{array}{l} \Gamma \vdash A \doteq A' \text{ type} \\ \Gamma, x.A \vdash B(x) \doteq B'(x) \text{ type} \end{array}}{\Gamma \vdash \Pi_{(x:A)} B(x) \doteq \Pi_{(x:A')} B'(x)} \begin{array}{l} \Pi\text{-jeq} \\ \Pi\text{-cong} \end{array}$$

Π -introdução

$$\frac{\Gamma, x:A \vdash b(x) : B(x)}{\Gamma \vdash \lambda x. b(x) : \prod_{(x:A)} B(x)} \quad \begin{array}{l} \lambda \\ \Pi\text{-I} \end{array}$$

cong:

$$\frac{\Gamma, x:A \vdash b(x) \doteq b'(x) : B(x)}{\Gamma \vdash \lambda x. b(x) \doteq \lambda x. b'(x) : \prod_{(x:A)} B(x)} \quad \begin{array}{l} \lambda\text{-Jeq} \\ \lambda\text{-cong} \end{array}$$

Π -eliminação

$$\frac{\Gamma \vdash f : \Pi_{(x:A)} B(x)}{\Gamma, x:A \vdash f x : B(x)} \begin{array}{l} \text{ev} \\ \text{app} \\ \Pi\text{-E} \end{array} *?$$

cong:

$$\frac{\Gamma \vdash f \doteq f' : \Pi_{(x:A)} B(x)}{\Gamma, x:A \vdash f x \doteq f' x : B(x)} \begin{array}{l} \text{ev-jeq} \\ \text{ev-cong} \end{array}$$

Π -computação

$$\frac{\beta \quad \Gamma, x:A \vdash b(x) : B(x)}{\Gamma, x:A \vdash (\lambda y. b(y)) x \doteq b(x) : B(x)} \Pi-\beta$$

$$\frac{\eta \quad \Gamma \vdash f : \Pi_{(x:A)} B(x)}{\Gamma \vdash \lambda x. f x \doteq f : \Pi_{(x:A)} B(x)} \Pi-\eta$$

Definições

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$D(\mathcal{H}_1, \dots, \mathcal{H}_n)$

mesma coisa escrita numa linha

D é uma derivação com raiz $\Gamma \vdash a : A$ com folhas abertas as $\mathcal{H}_1, \dots, \mathcal{H}_n$.

$\Gamma \vdash a : A$

$\Gamma \vdash c := a : A$

def[c]
-elem

$\Gamma \vdash A \text{ type}$

$\Gamma \vdash C := A \text{ type}$

def[C]
-type

Isso enriquece nossa teoria de tipos com uma nova constante C e com as regras seguintes:

$\mathcal{H}_1 \dots \mathcal{H}_n$

$\Gamma \vdash c : A$

C-elem

$\mathcal{H}_1 \dots \mathcal{H}_n$

$\Gamma \vdash c \doteq a : A$

C-def

$\mathcal{H}_1 \dots \mathcal{H}_n$

$\Gamma \vdash C \text{ type}$

C-type

$\mathcal{H}_1 \dots \mathcal{H}_n$

$\Gamma \vdash C \doteq A \text{ type}$

C-def

$(\rightarrow)$

$$\frac{\frac{\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, x.A \vdash B \text{ type}} \Pi}{\Gamma \vdash \Pi_{(x:A)}. B \text{ type}} \text{def } [\rightarrow]}{\Gamma \vdash A \rightarrow B := \Pi_{(x:A)} B \text{ type}} \text{-type}$$

kfam

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \rightarrow B \text{ type}} (\rightarrow) \text{-type}$$

+ cong

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \rightarrow B \doteq \Pi_{(x:A)} B \text{ type}} (\rightarrow) \text{-def}$$

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x:A \vdash b(x) : B}{\Gamma \vdash \lambda x. b(x) : A \rightarrow B} \quad \begin{array}{l} \lambda \\ (\rightarrow)\text{-I} \end{array} \quad + \text{ cong}$$

$$\frac{\Gamma \vdash f : A \rightarrow B}{\Gamma, x:A \vdash f x : B} \quad \begin{array}{l} \text{ev} \\ \text{app} \\ (\rightarrow)\text{-E} \end{array} \quad + \text{ cong}$$

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x:A \vdash b(x) : B}{\Gamma, x:A \vdash (\lambda y. b(y)) x \doteq b(x) : B} \quad \beta$$

$$\frac{\Gamma \vdash f : A \rightarrow B}{\Gamma \vdash \lambda x. f x \doteq f : A \rightarrow B} \quad \eta$$

identidade

$$\frac{\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x:A \vdash x:A}{\Gamma \vdash \lambda x.x : A \rightarrow A} \lambda (\rightarrow)\text{-I}}{\Gamma \vdash \text{id}_A := \lambda x.x : A \rightarrow A} \text{def[id]}_{\text{-elem}}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \text{id}_A : A \rightarrow A} \text{id-elem}$$

$\hookrightarrow$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \text{id}_A \doteq \lambda x.x : A \rightarrow A} \text{id-def}$$

currificação

nem temos (ainda)

$$\begin{aligned} (6) & : ((B \rightarrow C) \times (A \rightarrow B)) \rightarrow (A \rightarrow C) \\ & : (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \end{aligned}$$

$$\prod_{(x:A)} \prod_{(y:B(x))} C(x,y) \quad (x:A) \rightarrow (y:B(x)) \rightarrow C(x,y)$$

$$\prod_{(x,y:A)} C(x,y) \stackrel{\text{sup}}{\equiv} \prod_{(x:A)} \prod_{(y:A)} C(x,y)$$

Composição

$$\begin{array}{c}
 \frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma, f: B^A, x:A \vdash f x : B} \quad ? \\
 \frac{\Gamma, \dots^f, x:A \vdash f x : B \quad \Gamma, \dots, x:A, y:B \vdash g(y) : C}{\dots \vdash \dots} \quad w(\Gamma; g:C^B) \quad \text{sub}_{\text{elem}}(y:=fx) \\
 \frac{\Gamma, \dots \vdash C^A \text{ type} \quad \Gamma, g:C^B, f:B^A \vdash \lambda x. g(fx) : C^A}{\Gamma \vdash B^A \rightarrow C^A \text{ type} \quad \Gamma, g:C^B \vdash \lambda f \lambda x. g(fx) : B^A \rightarrow C^A} \quad \lambda \\
 \frac{\Gamma \vdash \lambda g. \lambda f. \lambda x. g(fx) : C^B \rightarrow (B^A \rightarrow C^A)}{\Gamma \vdash \text{comp} := \lambda g. \lambda f. \lambda x. g(fx) : C^B \rightarrow (B^A \rightarrow C^A)} \quad \lambda \quad \text{def}[\text{comp}]_{\text{elem}}
 \end{array}$$

Leis de identidade e composição

Associativity

$$(h \circ g) \circ f = h \circ (g \circ f) \quad ?$$

Unit:

$$\text{id}_B \circ f = f \quad ?$$

$$f \circ \text{id}_A = f \quad ?$$

constante

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, k:B \vdash \text{const}_k : A \rightarrow B} \text{const}_{\text{-elem}}$$

$$\text{const}_k := ?$$

$$\text{const}_c \circ f \doteq \text{const}_c : ?$$

$$g \circ \text{const}_c \doteq \text{const}_{(g \circ c)} : ?$$

flip ou swap

$$\Gamma \vdash A \text{ type}$$

$$\Gamma \vdash B \text{ type}$$

$$\Gamma, x:A, y:B \vdash C(x,y) \text{ type}$$

$$\Gamma \vdash \sigma : \prod_{x:A} \prod_{y:B} C(x,y) \rightarrow \prod_{y:B} \prod_{x:A} C(x,y) \text{ flip swap } \sigma$$

$$\sigma \circ \sigma \doteq ? : ?$$

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