P(r)oset

lec01

2025-08-18

Poset   carrier

$\mathcal{P} \equiv (P \; ; \; \leq)$

$P \; : \; \text{Set } \alpha$

$(\leq) \; : \; \underbrace{P \times P \to \text{Prop}}_{\text{BinRel }(P)}$

$\text{Set} \; : \; \text{Type} \to \text{Type}$

$\text{Set Int}, \;\; \text{Set Person}, \;\; \text{Set Nat} \; : \; \text{Type}$

Proset

pré-order

$(\leq)$-refl $\quad a \leq a \qquad\qquad \dfrac{}{a \leq a}\;\text{refl}$

$(\leq)$-trans $\quad a \leq b \;\&\; b \leq c \implies a \leq c \qquad\qquad \dfrac{a \leq b \quad b \leq c}{a \leq c}\;\text{trans}$

$(\leq)$-antis $\quad a \leq b \;\&\; b \leq a \implies a = b \qquad\qquad \dfrac{a \leq b \quad b \leq a}{a = b}\;\text{sym}$

Arr(Poset)

$$\mathcal{P} \xrightarrow{f} \mathcal{Q}$$

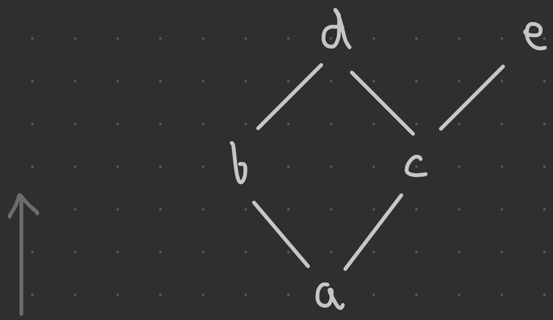
$$(P ; \leq_P) \qquad (Q ; \leq_Q)$$

$$f : P \to Q$$

f preserve a $(\leq)$

$$p \leq_P p' \implies f\,p \leq_Q f\,p'$$

Diagramas Hasse



isso determina um poset!

$$a \prec b \overset{\text{def}}{\iff} a \le b \ \& \ \overbrace{(\forall t)[a \le t \le b \Rightarrow t = a \text{ ou } t = b]}^{(\forall a \le t \le b)[t = a \text{ ou } t = b]}$$

coberto por

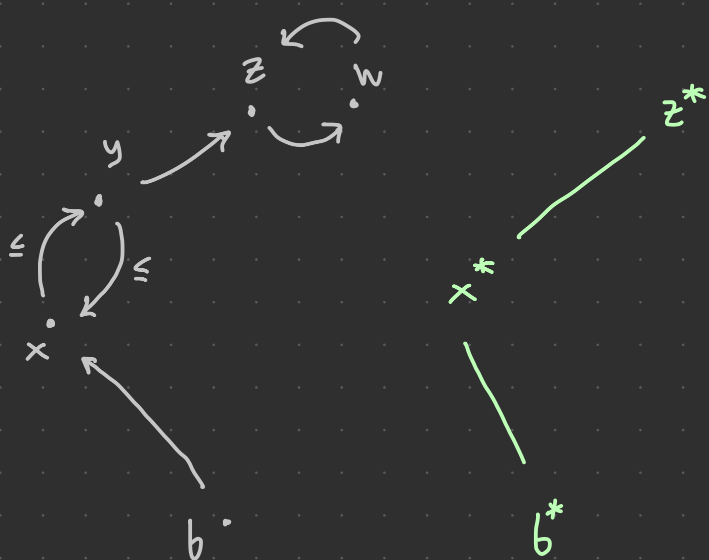
outra notação: $a < b$

HW: corrija a def de $(\prec)$

θ. Justificar o nome (pré-ordem)

Proset $\rightsquigarrow$ Poset

$(P ; \leq)$ $\qquad$ $(P^* ; \leq^*)$



$$a \sim b \overset{\text{def}}{\iff} a \leq b \ \& \ b \leq a$$

$(\sim)$ é uma relação de equivalência

$(\sim)$-refl
$(\sim)$-trans
$(\sim)$-sym

portanto

$$P/_{\sim} \equiv \{ [a]_{\sim} \mid a \in P \}$$

$$P/_{\sim} \equiv \left\{ \begin{array}{l} [b] = \{b\} \\ [x] = \{x, y\} \\ [y] = \{x, y\} \\ [z] = \{z, w\} \\ [w] = \{w, z\} \end{array} \right\}$$

$$X \leq^{*} Y \overset{\text{def}}{\iff} (\overset{\forall ?}{\exists} x \in X)(\overset{\forall ?}{\exists} y \in Y)[x \leq y]$$

vs:

escolha $x \in X$
escolha $y \in Y$

$$X \leq^{*} Y \underset{\text{def?}}{\iff} x \leq y$$

θ. Justificar o nome pré-metrica.

Bounds

$u \geqslant A \iff (\forall a \in A)[u \geq a]$

u upper bound of A

u melhor upper bound $\overset{def}{\iff}$ $u \geqslant A$ & $(\forall c \geqslant A)[u \leq c]$

lower $\quad \leq A \quad \leq A \quad \geqslant c$

Garantias

P poset

lattice	finite subsets	have joins
complete lattice	subsets	have joins
dcpo	directed sets	have joins
ω-cpo	ω-chains	have joins

D directed $\overset{\text{def}}{\iff} (\forall u, v \in D)(\exists d \in D)[\, d \geq \{u, v\} \,]$

$$\mathcal{L} \equiv (L; \leq)$$

($\leq$)-fin lub

($\leq$)-fin glb

$$a \vee b \equiv \text{lub}\,\{a, b\}$$

$$a \wedge b \equiv \text{glb}\,\{a, b\}$$

$$\bot \equiv \text{lub}\,\emptyset$$

$$\top \equiv \text{glb}\,\emptyset$$

$$\mathcal{L} \equiv (L;\ \overset{\text{join}}{\underset{2}{\vee}},\ \overset{\text{meet}}{\underset{2}{\wedge}},\ \overset{0}{\underset{0}{\bot}},\ \overset{\wedge}{\underset{0}{\top}})$$

semilattice

($\vee$)-ass

($\vee$)-com

($\vee$)-idem $\quad x \vee x = x$

($\bot$)-id-($\vee$)

($\vee, \wedge$)-abs $\quad (a \vee b) \wedge a = a$

($\top$)-ann-($\vee$) $\quad x \vee \top = \top$

$\bot \quad \wedge \quad x \wedge \bot = \bot$

$$a \leq b \iff a \vee b = b$$

$$\iff a \wedge b = a$$

$$A \leq B \overset{def}{\Longleftrightarrow} A \text{ true} \vdash B \text{ true}$$

$$\frac{}{a \leq a \vee b}\ I_1 \qquad \frac{}{b \leq a \vee b}\ I_2$$

$a \vee b$ é um u.b. do $\{a, b\}$

$$\frac{a \leq c \qquad b \leq c}{a \vee b \leq c}\ E$$

$A \text{ true} \vdash B \text{ true}$

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$$

$$\dfrac{\dfrac{\dfrac{}{x \wedge y \leq x} \qquad x \wedge y \leq (y \vee z)}{x \wedge y \leq x \wedge (y \vee z)} \qquad \dfrac{\dfrac{}{x \wedge z \leq x} \qquad x \wedge z \leq y \vee z}{x \wedge z \leq x \wedge (y \vee z)}}{(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)}$$

Distributivity & Modularity

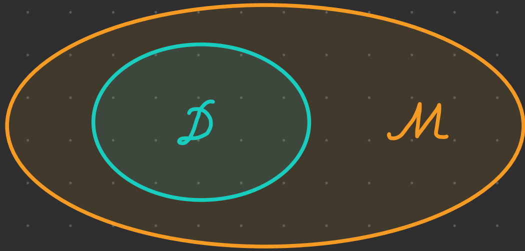
Todo reticulado é "meio" distributivo & "meio" modular:

- Θ. $d \wedge (a \vee b) \geq (d \wedge a) \vee (d \wedge b)$ [safe-distr-(∧),(∨)]

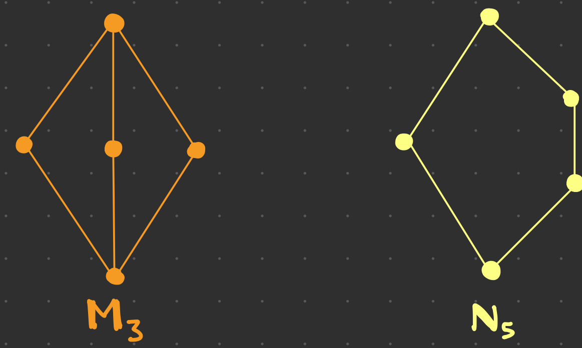
- Θ. $a \leq b \implies a \vee (x \wedge b) \leq (a \vee x) \wedge b$ [safe-modular]

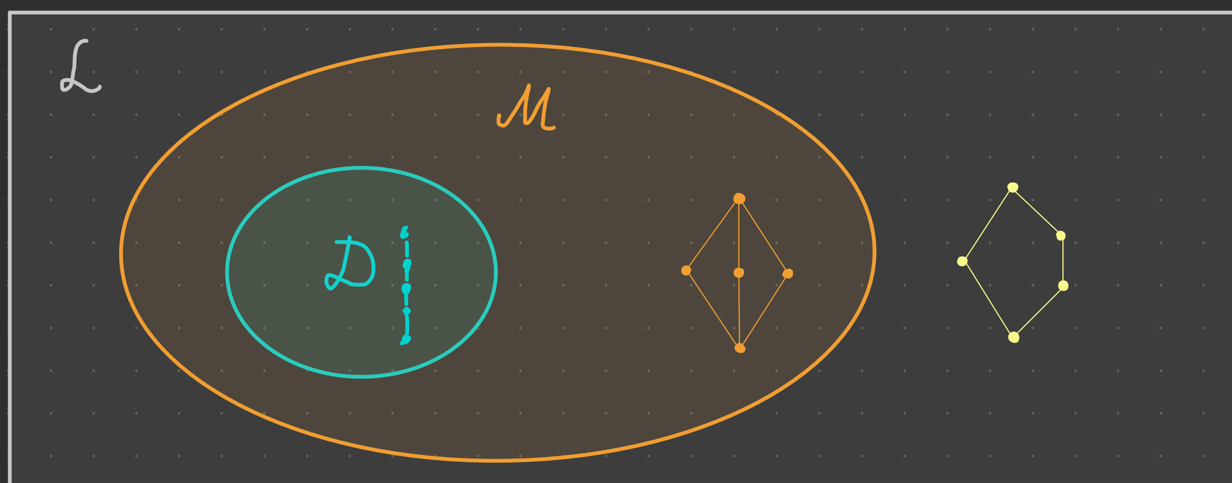
- Θ. $\underbrace{(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)}_{\text{join of meets}} \leq \underbrace{(a \vee b) \wedge (b \vee c) \wedge (c \vee a)}_{\text{meet of joins}}$

- Θ. distr $\Rightarrow$ modular

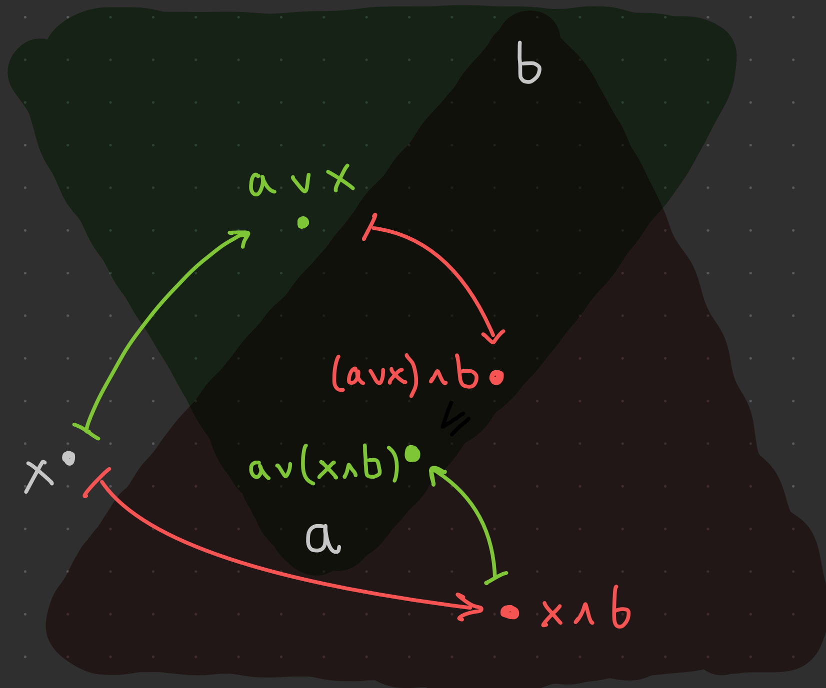


Θ^2.

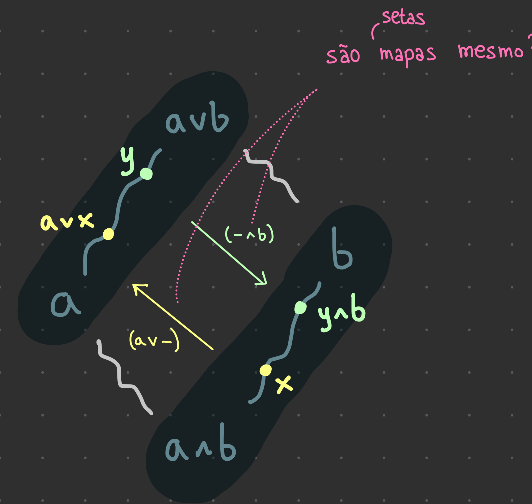




Modularity



$(a \vee -)$ $(- \wedge b)$



Ups & downs

$\uparrow A \overset{\text{def}}{\equiv} \{ x \in P \mid (\exists a \in A)[a \leq x] \}$ $\qquad$ $\uparrow a \overset{\text{def}}{\equiv} \uparrow\{a\}$

$\downarrow A$ ... $\qquad$ $\downarrow a$ ...

fechado para cima

(ou: upset)

A upper set $\overset{\text{def}}{\iff} \underbrace{(\forall a \in A)\overbrace{(\forall x \in P)[\, a \leq x \Rightarrow x \in A \,]}^{(\uparrow a \,\subseteq\, A)}}_{A \,=\, \uparrow A}$

(ou: downset)

A lower set ...

$(a, b) \overset{\text{def}}{\equiv} \{ x \mid a < x < b \}$

$($ $\quad$ $]$ $\qquad$ $<$ $\quad$ $\leq$

$[$ $\quad$ $)$ $\qquad$ $\leq$ $\quad$ $<$

$[$ $\quad$ $]$ $\qquad$ $\leq$ $\quad$ $\leq$

Ideais de reticulados

I a-ideal $\iff$ I sub-$(\vee)$-semilattice de R

&

I absorve os meets $\iff$ $I \wedge R \subseteq I$

I o-ideal $\iff$ I downset

&

I habitado

&

I $(\vee)$-fechado

Θ. I a-ideal $\Leftrightarrow$ I o-ideal

$(o \Rightarrow a)$: I hab $\rightsquigarrow$ $i \in I$ $\rightsquigarrow$ $\bot \in I$

I $(\vee)$-fechado

$i \wedge r \leq i \in I \rightsquigarrow i \wedge r \in I$

$(a \Rightarrow o)$: ?

Prime ideals

$$I \text{ primo} \overset{\text{def}}{\iff} \underbrace{I \text{ próprio}}^{I \neq L} \;\&\; (\forall a,b \in L)[a \wedge b \in I \Rightarrow a \in I \text{ ou } b \in I]$$