Coleções

Conjuntos (Set) & tuplas

lec01

2025-03-17

$2 + 3 : \mathrm{Int}$

$\shortparallel$

$5 : \mathrm{Int}$

igualdade (extensional)

$\{2, 3\} : \mathrm{Set}$ $\qquad$ $\mathrm{Set} : \mathrm{Type}$

$2 : \mathrm{Int}$ $\qquad$ $\mathrm{Int} : \mathrm{Type}$

$3 : \mathrm{Int}$

implícitas pela precedência sintáctica do (×) contra o (→)

cross

$(+) : (\mathrm{Int} \times \mathrm{Int}) \to \mathrm{Int}$

$\_+\_$:

tuplas

$(2, 3) : \mathrm{Int} \times \mathrm{Int}$

$(5, \text{θάνος}) : \mathrm{Int} \times \mathrm{Person}$

$\{\theta, \text{lury}\} : \mathrm{Set}$

$(\times) : \mathrm{Type} \times \mathrm{Type} \to \mathrm{Type}$

brrr...

$\_\cup\_ : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$

açucar sintáctico

$2 + 3$ notação infixa

$\equiv$ igualdade intensional

$(+)\ (2,3)$ notação prefixa

$\boxed{)\ \sqrt{}\ S\ +}$

lixo

$$\left.\dfrac{(+) : \mathrm{Int} \times \mathrm{Int} \to \mathrm{Int} \qquad \dfrac{2 : \mathrm{Int} \qquad 3 : \mathrm{Int}}{(2,3) : \mathrm{Int} \times \mathrm{Int}}}{(+) \sqcup (2,3) \ : \ \mathrm{Int}}\right\}\ \text{arvore de inferência}$$

aplicação de função

$f : \mathrm{Int} \to \mathrm{Int}$ $\qquad f(3) \qquad f \sqcup 3$

$g : \mathrm{Int} \times \mathrm{Int} \to \mathrm{Int}$ $\qquad g(2,3) \qquad g \sqcup (2,3)$

argumentos

$\{2, 3\} : \text{Set} ␣ \text{Int}$

$\{\Theta, \text{lury}\} : \text{Set} ␣ \text{Person}$

$\text{Set} : \text{Type} \to \text{Type}$

$\text{Int} : \text{Type}$

$\_ \cup_{\text{Int}} \_ : \text{Set Int} \times \text{Set Int} \to \text{Set Int}$

$\_ \cup_{\text{Person}} \_ : \text{Set Person} \times \text{Set Person} \to \text{Set Person}$

polimorfismo

$\_ \cup_{\alpha} \_ : \text{Set } \alpha \times \text{Set } \alpha \to \text{Set } \alpha$

$\{2, 3\} \cup_{\text{Int}} \{3, 5\} : \text{Set Int}$

$\{\Theta, I\} \cup_{\text{Person}} \{I, Y\} : \text{Set Person}$

$\{2, 3\} \cup_{?} \{\Theta, I\}$ Type error!

Operações de conjuntos de α

$\_ \cup_\alpha \_ \; : \; \mathrm{Set}\,\alpha \times \mathrm{Set}\,\alpha \longrightarrow \mathrm{Set}\,\alpha$

$\_ \cap_\alpha \_$

$\_ \setminus_\alpha \_$

$\_ \triangle_\alpha \_$

$\wp_\alpha \_ \; : \; ?$

$\bigcup_\alpha \_$

$\bigcap_\alpha \_$

Funções

lec02

2025-03-21

domínio

codomínio

$$f \;:\; \alpha \to \beta$$

intensão

$$f\,x : \beta \longleftarrow \boxed{{}^{\beta}\; f \;{}^{\alpha}} \longleftarrow x : \alpha$$

black box

$$f\ a_1 \overset{\text{def}}{\equiv} \dots$$
$$f\ a_2 \overset{\text{def}}{\equiv} \dots$$
$$\vdots$$

aplicação de função

determinei como f comporta para cada possível pergunta/observação que sua interface permite;

portanto determinei a f

u␣v␣w ?

igualdade (entre funções)

$$\boxed{f =_{\alpha\to\beta} g \overset{\text{def}}{\iff} (\forall a : \alpha)[\, f\,a =_{\beta} g\,a \,]}$$

igualdade extensional (extensionalidade de funções)

$(=_{Nat})$ vs $(=_{Real})$

$$n \overset{?}{=}_{Nat} m$$

n vezes $\left\{ \begin{matrix} S \\ S \\ \vdots \\ S \end{matrix} \right.$ O $\qquad$ $\left. \begin{matrix} S \\ S \\ \vdots \\ S \end{matrix} \right\}$ m vezes O

$\cancel{S}$ $\qquad$ $\cancel{S}$

n-1 vezes $\left\{ \begin{matrix} S \\ \vdots \\ S \end{matrix} \right.$ O $\qquad$ $\left. \begin{matrix} S \\ \vdots \\ S \end{matrix} \right\}$ m-1 vezes O

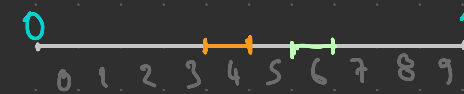
$\vdots$

este processo termina em uma quantidade finita de passos

$$s \overset{?}{=}_{Real} t$$

4 $\qquad$ 6

$\vdots$ $\qquad$ $\vdots$

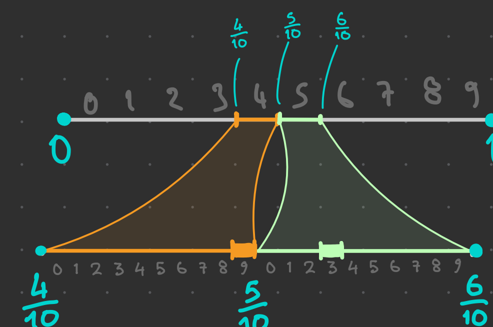


$$s \overset{?}{=}_{Real} t$$

4 $\qquad$ 5

9 $\qquad$ 3

$\vdots$ $\qquad$ $\vdots$



$(=_{Real})$ vs $(\neq_{Real})$?

Como definir operações de conjuntos

$\_ \cup_\alpha \_ \;:\; \mathrm{Set}\ \alpha \times \mathrm{Set}\ \alpha \to \mathrm{Set}\ \alpha$

$X \cup Y \overset{\mathrm{def}}{\equiv} \ldots ? \ldots : \mathrm{Set}\ \alpha$

preciso saber como definir conjuntos (veja próximo quadro)

$\underset{\alpha}{a} \in \underbrace{X \cup Y}_{\mathrm{Set}\ \alpha} \overset{\mathrm{def}}{\iff} \underbrace{a \in X \overset{\vee}{\text{ou}} a \in Y}_{\mathrm{Prop}}$

notação set-builder

set comprehension

o conjunto de todos os ☐ tais que ☐ (silêncio)

$X \cup Y \overset{\mathrm{def}}{\equiv} \{\, \boxed{a : \alpha} \mid \boxed{a \in X \text{ ou } a \in Y} \,\}$

Como definir conjuntos

Como determinar um conjunto (i.e. um habitante do tipo $\mathrm{Set}\ \alpha$)

$A : \mathrm{Set}\ \alpha$

$\_ \in_\alpha \_ : \alpha \times \mathrm{Set}\ \alpha \to \mathrm{Prop}$

$x \in A : \cancel{\mathrm{Bool}}\ \mathrm{Prop}$

isso determina o conjunto A

$x \in A \overset{\text{def}}{\iff} \underbrace{\ldots x \ldots}_{\varphi(x)}$

$$\mathrm{Set}\ \alpha \overset{?}{\equiv} \alpha \to \mathrm{Prop}$$

$A \overset{\text{def}}{\equiv} \{ a : \alpha \mid \varphi(a) \}$

$\varphi \sqcup a : \mathrm{Prop}$

$\varphi : \alpha \to \mathrm{Prop}$

$\{ a : \alpha \mid \varphi(x) \}$

HW:

Quantos membros tem?

Variáveis livres e ligadas (binding)

$$\{a : \alpha \mid a \in X \text{ ou } a \in Y\}$$

procura ocorrências livres de a para ligar

estavam livres aqui

$$a \in X \text{ ou } a \in Y$$

$$\{a : \alpha \mid a \in X \text{ ou } a \in \{a : \alpha \mid a \in Y\}\}$$

$$\{\bullet : \alpha \mid \bullet \in X \text{ ou } \bullet \in \{\bullet : \alpha \mid \bullet \in Y\}\}$$

Como definir **relações** entre conjuntos

$$\_ \subseteq_\alpha \_ \; : \; \mathrm{Set}\ \alpha \times \mathrm{Set}\ \alpha \to \mathrm{Prop}$$

'a' é o melhor nome aqui

$$X \subseteq_\alpha Y \overset{\text{def}}{\iff} (\forall a : \alpha)[\, a \in_\alpha X \Rightarrow a \in_\alpha Y \,]$$

$$\Big\Updownarrow \text{sug}$$

HW: defina esse açucar sintáctico!
...E sobre o $\exists$?

$$(\forall a \in X)[\, a \in Y \,]$$

$$(\forall x \in X)[\, x \in Y \,]$$

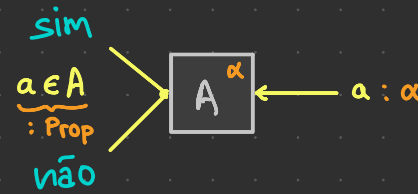
agora 'x' é o melhor nome

$$X \subsetneq_? Y \overset{\text{def}}{\iff} X \subseteq_? Y \;\&\; X \neq_? Y$$

? o que tá implícito aqui?

> Cuidado: $(\notin) \neq (\subsetneq)$

Igualdade



$$X =_{\mathsf{Set}\ \alpha} Y \quad \overset{\text{def}}{\iff} \quad (\forall a : \alpha)[\, a \in X \iff a \in Y \,] \qquad : \mathsf{Cmd}$$

Θ. $$X = Y \iff X \subseteq Y \;\&\; Y \subseteq X \qquad : \mathsf{Prop}$$

$$X \overset{\text{def}}{\equiv} \{ a : \mathrm{Int} \mid a = 2 \}$$

$$Y \overset{\text{def}}{\equiv} \{ a : \mathrm{Int} \mid a \underset{a \geq 0}{\text{ primo }} \& \ a \text{ par} \}$$

$$\boxed{X = Y \ \text{ mas } \ X \not\equiv Y}$$

Como construir vs. como usar

aquele/o conjunto de todos os ...

$$\{ a : \alpha \mid \varphi(a) \}$$

$$SSSO + SSO \xrightarrow{\text{passo computacional}} S(SSO + SO)$$

$$x \in \{ a : \alpha \mid \varphi(a) \} \xrightarrow{\beta} \varphi(x)$$

Exemplo.

$$x \in \{ a : Int \mid \overbrace{a \text{ primo} \ \& \ a < 5}^{\varphi(a)} \} \xrightarrow{\beta} x \text{ primo} \ \& \ x < 5$$

λ notação (fun builder)

aquela função que recebendo um $\boxed{}$ retorna $\boxed{}$

$$\lambda\ \boxed{x : \alpha}\ .\ \boxed{\tau}$$

constrói uma função anônima

taf (tau)

possivelmente há ocorrências (livres) de x aqui

Compare:

o conjunto de todos os $\boxed{}$ tais que $\boxed{}$ (silêncio)

constrói um conjunto anônimo

$$\{\ \boxed{a : \alpha}\ |\ \boxed{a \in X \text{ ou } a \in Y}\ \}$$

β

$$\lambda x .\ x + 2 \quad : \quad Int \to Int$$

$$\lambda x .\ x + 2 \text{␣} 3 \ \stackrel{?}{=}\ \text{TYPE ERROR}$$

$$\underbrace{(\lambda x .\ x + 2) \text{␣} 3}_{\beta\text{-redex}} \ \stackrel{?}{=}\ 5 \qquad (x+2)[x := 3] \equiv 3+2$$

reducible expr

$$\Big\downarrow \beta$$

$$3 + 2$$

$$(\lambda x : \alpha .\ \tau) \text{␣} a$$

$$\Big\downarrow \beta$$

$$\tau[x := a]$$

Substituição

$\tau[x := a] \overset{\text{def}}{\equiv}$ a expressão obtida substituindo <u>todas</u> as ocorrências livres de x por a

Exemplos:

$$\left(t + \int_0^1 f(t)\,dt\right)[t := 5] \equiv 5 + \int_0^1 f(t)\,dt$$

$$\left(t \cdot x + \int_0^x f(x)\,dx\right)[t := 2;\ x := 5] \equiv 2 \cdot 5 + \int_0^5 f(x)\,dx$$

Empty , Singleton

$\text{Empty} : \text{Set } \alpha \to \text{Prop}$

$$\underset{\text{Set } \alpha}{\text{Empty } X \;:} \overset{\text{def}}{\iff} (\forall a : \alpha)[\, a \notin X \,]$$

$$\text{Singleton } X \overset{\text{def}}{\iff} (\exists! a : \alpha)[\, a \in X \,]$$

$$\text{Subsingleton } X \overset{\text{def}}{\iff} (\forall u, v \in X)[\, u = v \,]$$

$$\text{Universal } X \overset{\text{def}}{\iff} (\forall a . \alpha)[\, a \in X \,]$$

Existência & unicidade

pelo menos um

$$(\exists!\, a : \alpha)[\,\varphi(a)\,] \overset{\text{sug}}{\iff} \begin{array}{l}(\exists a : \alpha)[\,\varphi(a)\,] \\ \quad \& \\ (\forall u, v : \alpha)[\,\varphi(u)\ \&\ \varphi(v) \Rightarrow u = v\,]\end{array}$$

no máximo um (proof by fight club)

Θ. Existência & "Unicidade" de vazio $\rightsquigarrow$ $\emptyset_\alpha$

$\emptyset \in \emptyset$

$A \subseteq A$

$\emptyset \in \emptyset$

$\emptyset \in A$

$\vdots$

Investigue!

Operações de conjuntos de α

lec04

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$\_\cup_\alpha\_ \ : \ \mathrm{Set}\ \alpha \times \mathrm{Set}\ \alpha \to \mathrm{Set}\ \alpha$

$\_\cap_\alpha\_ \ :$

$$a \in X \cap Y \overset{\text{def}}{\iff} a \in X \ \& \ a \in Y$$

$$X \cap Y \overset{\text{def}}{\equiv} \{ a : \alpha \mid a \in X \ \& \ a \in Y \}$$

$\_\setminus_\alpha\_ \ : \ \mathrm{Set}\ \alpha \times \mathrm{Set}\ \alpha \to \mathrm{Set}\ \alpha$

$$X \setminus Y \overset{\text{def}}{\equiv} \{ a : \alpha \mid a \in X \ \& \ a \notin Y \}$$

$\_\triangle_\alpha\_ \ : \ \mathrm{Set}\ \alpha \times \mathrm{Set}\ \alpha \to \mathrm{Set}\ \alpha$

$$X \triangle Y \overset{\text{def}}{\equiv} \begin{cases} (X \setminus Y) \cup (Y \setminus X) \\ \qquad ?\!= \\ (X \cup Y) \setminus (X \cap Y) \end{cases}$$

$\emptyset_\alpha : \mathrm{Set}\ \alpha$ $\qquad (+) \rightsquigarrow \Sigma \qquad (\cup) \rightsquigarrow \bigcup$

$\mathcal{U}_\alpha : \mathrm{Set}\ \alpha$ $\qquad (\cdot) \rightsquigarrow \Pi \qquad (\cap) \rightsquigarrow \bigcap$

powerset («conjunto potência», «conjunto das partes»)

$\wp_\alpha\ \_ : \mathrm{Set}\ \alpha \to \mathrm{Set}\ (\mathrm{Set}\ \alpha)$ $\qquad \wp\{1,2\} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$A \in \wp X \overset{\text{def}}{\iff} A \subseteq X$

$\bigcup_\alpha\ \_ : \mathrm{Set}\ (\mathrm{Set}\ \alpha) \to \mathrm{Set}\ \alpha$ $\qquad \bigcup\{\{1,2\},\{2,3\},\{5,7\}\}$

?

$= \{1,2,3,5,7\}$

$\bigcap_\alpha\ \_ : \mathrm{Set}\ (\mathrm{Set}\ \alpha) \to \mathrm{Set}\ \alpha$ $\qquad \bigcap\{\{1,2,3\},\{2,3,4\}, \mathrm{Primos}\}$

?

$= \{2,3\}$

Operações "se anulam"?

$$
\begin{tikzcd}
X \arrow[r, mapsto, "\wp"] \arrow[dr, equal, "?"'] & \wp X \arrow[d, mapsto, "\bigcup"] \\
& \bigcup \wp X
\end{tikzcd}
\qquad \bigcap ? \qquad
\begin{tikzcd}
X \arrow[r, mapsto, "\bigcup"] \arrow[dr, equal, "?"'] & \bigcup X \arrow[d, mapsto, "\wp"] \\
& \wp \bigcup X
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Set}\ \alpha \arrow[rr, bend left, "\wp"] & & \mathrm{Set}\ (\mathrm{Set}\ \alpha) \arrow[ll, "\bigcap"'] \arrow[ll, bend left, "\bigcup"]
\end{tikzcd}
$$

Set-builder mais poderoso?

$$Z \stackrel{\text{def}}{\equiv} \{ \underbrace{x^2+1}_{\text{não é um Var (nome de variável)}} : \text{Int} \mid \overbrace{x \in \{1,2\}}^{\text{gerador}} \}$$

não! (Mas é conveniente)

$$\equiv \{ 1^2+1, 2^2+1 \} = \{2, 5\}$$

$$Z' \stackrel{\text{def}}{\equiv} \{ x^2+1 \mid x \in \{1,2\}, x+4 \text{ primo} \}$$

$$\equiv \{ 1^2+1 \} = \{2\}$$

$$Z'' \stackrel{\text{def}}{\equiv} \{ x^2+y \mid x \in \{1,2\}, y \in \{1,2,3\}, x+y \text{ prime} \}$$

$$\{ 1^2+1, \ldots, 2^2+3 \}$$

$$y \in \{ \underbrace{\tau(x_1, \dots, x_n)}_{\text{expressão}} \mid x_1 \in G_1, \dots, x_n \in G_n \}$$

$$\Updownarrow \text{def}$$

$$(\exists x_1 \in G_1) \cdots (\exists x_n \in G_n)[\, y = \tau(x_1, \dots, x_n) \,]$$

$$y \in \{ \underbrace{\tau(x_1, \dots, x_n)}_{\text{expressão}} \mid x_1 \in G_1, \dots, x_n \in G_n, \overbrace{\varphi(x_1, \dots, x_n)}^{\text{Prop}} \}$$

$$\Updownarrow \text{def}$$

?

$\{ a \in A \mid \dots \}$?

lec05

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$\wp_\alpha \_ \quad : \quad \mathsf{Set}\ \alpha \to \mathsf{Set}\ (\mathsf{Set}\ \alpha)$

$$A \in \wp X \overset{\text{def}}{\iff} A \subseteq X$$

$\bigcup_\alpha \_ \quad : \quad \mathsf{Set}\ (\mathsf{Set}\ \alpha) \to \mathsf{Set}\ \alpha$

♡ os a que petencem àlgum dos membros de $\mathcal{A}$.

$$a \in \bigcup \mathcal{A} \overset{\text{def}}{\iff} (\exists A \in \mathcal{A})[\, a \in A \,] \iff (\exists A)[\, A \in \mathcal{A} \;\&\; a \in A \,]$$

$\bigcap_\alpha \_ \quad : \quad \mathsf{Set}\ (\mathsf{Set}\ \alpha) \to \mathsf{Set}\ \alpha$

♡ os a que petencem a todos os membros de $\mathcal{A}$.

$$a \in \bigcap \mathcal{A} \overset{\text{def}}{\iff} (\forall A \in \mathcal{A})[\, a \in A \,] \iff (\forall A)[\, A \in \mathcal{A} \Rightarrow a \in A \,]$$

$\mathcal{A}$: Set (Set Int)

$2 \in \bigcap \{ \{1,2,3\}, \{2,3,4\}, \text{Primos} \}$

$\Updownarrow$

$(\forall A : \text{Set Int})[\, A \in \{ \{1,2,3\}, \{2,3,4\}, \text{Primos} \} \ \& \ 2 \in A \,]$

(Assim $2 \notin \bigcap \{ \{1,2,3\}, \{2,3,4\}, \text{Primes} \}$

2-Tuplas

$$\frac{a : \alpha \qquad\qquad b : \beta}{\langle a , b \rangle_{\alpha,\beta} : \alpha \times \beta}$$

$$\begin{array}{lllll} (w)_1 & \pi_1\, w & \text{outl } w & w.1 & w \cdot l : \alpha \\ (w)_2 & \pi_2\, w & \text{outr } w & w.2 & w \cdot r : \beta \end{array} \longleftarrow \boxed{\begin{array}{c} \alpha \times \beta \\ w \end{array}}$$

β

$$\langle a , b \rangle \cdot l \overset{\beta}{\rightsquigarrow} a$$

$$\langle a , b \rangle \cdot r \overset{\beta}{\rightsquigarrow} b$$

Igualdade

$w, w' : \alpha \times \beta$

$$w =_{\alpha\times\beta} w' \iff w \cdot l =_\alpha w' \cdot l \ \ \& \ \ w \cdot r =_\beta w' \cdot r$$

$\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$

$\{1, 2\} = \{2, 1\}$

$\langle 1, 2 \rangle \overset{?}{=} \langle 2, 2, 1, 2 \rangle$

TYPE ERROR

$\{1, 2\} = \{2, 2, 1, 2\}$

$x \in \{1, 2\} \iff x = 1$ ou $x = 2$

3-Tuplas

$$\frac{a : \alpha \qquad b : \beta \qquad c : \gamma}{\langle a, b, c \rangle_{\alpha,\beta,\gamma} : \alpha \times \beta \times \gamma}$$

bla bla bla ...

n-Tuplas

$$\frac{a_1 : \alpha_1 \qquad \dots \qquad a_n : \alpha_n}{\langle a_1, \dots, a_n \rangle_{\alpha_1,\dots,\alpha_n} : \alpha_1 \times \dots \times \alpha_n}$$

bla bla bla ...

$n := 1$?

$n := 0$?

Implementação de tipo: 3-tupla como 2-tupla

$$\overset{t}{w} : \alpha \times \beta \times \gamma \xrightarrow{\text{tradução}} ? : (\alpha \times \beta) \times \gamma$$

$$\text{trad} : (\alpha \times \beta \times \gamma) \to ((\alpha \times \beta) \times \gamma)$$

$$\text{trad} \ \langle a, b, c \rangle \overset{\text{def}}{\equiv} \langle \langle a, b \rangle, c \rangle$$

pattern matching

$a : \alpha$
$b : \beta$
$c : \gamma$

$$\text{trad} \ t \overset{\text{def}}{\equiv} \langle \langle t\ 1, t\ 2 \rangle, t.3 \rangle$$

$t : \alpha \times \beta \times \gamma$

$$((\mathrm{trad}\ t)\cdot\ell)\cdot\ell \overset{\mathrm{def}}{\equiv} t.1 : \alpha$$

$$((\mathrm{trad}\ t)\cdot\ell)\cdot r \overset{\mathrm{def}}{\equiv} t.2 : \beta$$

$$(\mathrm{trad}\ t)\cdot\ell \overset{\mathrm{def}}{\equiv} \overbrace{\langle t.1, t.2\rangle}^{\alpha\times\beta}$$

$$(\mathrm{trad}\ t)\cdot r \overset{\mathrm{def}}{\equiv} t.3 : \gamma$$

$$t.1 \longmapsto ((\mathrm{trad}\ t)\cdot\ell)\cdot\ell$$

$$t.2 \longmapsto ((\mathrm{trad}\ t)\cdot\ell)\cdot r$$

$$t.3 \longmapsto (\mathrm{trad}\ t)\cdot r$$

Finalmente

$$\langle a, b, c\rangle_{\alpha,\beta,\gamma} \overset{\mathrm{sug}}{\equiv} \langle\langle a, b\rangle_{\alpha,\beta}, c\rangle_{\alpha\times\beta,\gamma}$$

$$t.1 \overset{\mathrm{sug}}{\equiv} ((\mathrm{trad}\ t)\cdot\ell)\cdot\ell$$

$$t.2 \overset{\mathrm{sug}}{\equiv} ((\mathrm{trad}\ t)\cdot\ell)\cdot r$$

$$t.3 \overset{\mathrm{sug}}{\equiv} (\mathrm{trad}\ t)\cdot r$$

Implementar 2-tuplas $\alpha \times \alpha$

Como não implementar: $\langle a_1, a_2 \rangle \overset{\text{trad}}{\equiv} \{a_1, a_2\}$

Por quê?: $\langle 0,1 \rangle = \{0,1\} = \{1,0\} = \langle 1,0 \rangle$

Testando tentativas de implementar

lec06

2025-04-04

$$\langle x, y\rangle \stackrel{?}{\equiv} \{x, \{y\}\}$$

Tipado (tipos estáticos)

static typing

$\mathrm{Set}\ \alpha : \mathrm{Type}$ $(\mathrm{Set} : \mathrm{Type} \to \mathrm{Type})$

Não funciona:

$$\langle 1, 2\rangle \equiv \boxed{\{1, \{2\}\}}$$

mal tipado

Un(i)tipado (tipos dinâmicos)

dynamic typing

$\mathrm{Set} : \mathrm{Type}$

$\langle 0,1\rangle \neq \langle 1,0\rangle$ (como deveria)

mas isso não é suficiente!

$$\pi_1(\{x, \{y\}\}) \stackrel{\text{def}}{\equiv} x$$

não é um construtor

```
def n00b(x + y):
    return 2x + y
```

tentativa miserável de pattern matching

$$\pi_1(s) \stackrel{\text{def}}{\equiv} \ldots?$$

$(\times)$: formar, construir, usar, computar
(Introduzir) (Eliminar) (β, η)

$$\frac{\alpha \ \mathsf{Type} \qquad \beta \ \mathsf{Type}}{\alpha \times \beta \ \mathsf{Type}} \ (F)$$

$$\frac{a : \alpha \qquad b : \beta}{\langle a, b \rangle : \alpha \times \beta} \ (I)$$

como construir habitantes (construtor)

$$\frac{w : \alpha \times \beta}{w \cdot l : \alpha} \ (E) \qquad \frac{w : \alpha \times \beta}{w \cdot r : \beta} \ (E)$$

como usar habitantes (interface)

$$\langle a, b \rangle \cdot l = a$$
$$\langle a, b \rangle \cdot r = b$$

(β) como computar (η)

$$\langle w \cdot l, w \cdot r \rangle = w$$

construir e logo depois usar (extrair)

unicidade, sem lixo, ...

extrair (usar) e logo depois construir

(β) & (η) para Fun e Set

pode ter ocorrências de x

(β) - Fun $\quad (\lambda x.\, b) ␣ a = b[x := a]$

(η) - Fun $\quad \lambda x.\, f ␣ x = f$

(β) - Set $\quad a \in \{x \mid \varphi(x)\} \iff \varphi(a)$

(η) - Set $\quad \{x \mid x \in A\} = A$

Unit

$$\frac{\text{nada}}{\mathsf{Unit}\ \mathsf{type}}\ (F) \qquad \frac{}{\star : \mathsf{Unit}}\ (I) \qquad \text{nenhuma}\ (E)$$

$\mathbb{1}$ $\mathbb{1}$

☺

()

...?

Bool

$$\frac{}{\mathsf{Bool}\ \mathsf{type}}\ (F) \qquad \frac{}{\mathit{ff} : \mathsf{Bool}}\ (I) \qquad \frac{}{\mathit{tt} : \mathsf{Bool}}\ (I)$$

false

true

?...

Empty

$$\frac{}{\mathsf{Empty}\ \mathsf{type}}\ (F) \qquad \text{nenhuma}\ (I) \qquad \frac{e : \mathsf{Empty}}{\mathsf{boom}\ e : \alpha}\ (E)$$

0 $\mathbb{0}$

$y : C$ $\quad n : C$

abort

...?

(void?)

```
int f (void)
    return 42;
```

```
void g (int x)
    return;
```

$$\frac{f : \overset{\text{Unit}}{\overset{\cancel{\text{Empty}}}{\text{void}}} \to \text{Int} \qquad \overset{()}{a} : \overset{\text{Unit}}{\text{void}}}{f\ a : \text{Int}}$$

$$f\ ()$$

$$\frac{g : \text{Int} \to \overset{\text{Unit}}{\overset{\cancel{\text{Empty}}}{\text{void}}} \qquad n : \text{Int}}{g\ n : \underset{\text{Unit}}{\underset{\cancel{\text{Empty}}}{\text{void}}}}$$

$f\ ()$ — um argumento: o ()

aplicação

$\boxed{f(\ldots)}$ — muitos programadores pensam assim

(nenhum argumento)

Composição

$$\begin{tikzcd}
\alpha \arrow[r, "f"] \arrow[rr, bend right, dotted, "f;g"' above, "g \circ f"' below] & \beta \arrow[r, "g"] & \gamma
\end{tikzcd}$$

$$(\circ)_{\alpha\to\beta\to\gamma} : (\beta \to \gamma) \times (\alpha \to \beta) \longrightarrow (\alpha \to \gamma)$$

$g \circ_{\alpha\to\beta\to\gamma} f \overset{\text{def}}{\equiv} \lambda a.\, g\,(f\ a)$ **ou** $(g \circ_{\alpha\to\beta\to\gamma} f)\ a \overset{\text{def}}{\equiv} g\,(f\ a)$

Θ. (∘)-Assoc.

θ. (∘)-Assoc.

lec07

2025-04-07

$$(\forall\ \alpha \xrightarrow{h} \beta \xrightarrow{g} \gamma \xrightarrow{f} \delta)\left[(f\circ g)\circ h = f\circ(g\circ h)\right]$$

$$(\forall \alpha,\beta,\gamma,\delta : \mathrm{Type})(\forall h:\alpha\to\beta)(\forall g:\beta\to\gamma)(\forall f:\gamma\to\delta)$$

Sejam $\alpha \xrightarrow{h} \beta \xrightarrow{g} \gamma \xrightarrow{f} \delta$.

Demonstre $(f\circ g)\circ h =_{\alpha\to\delta} f\circ(g\circ h)$.

$$\dfrac{h:\alpha\to\beta \qquad \dfrac{g:\beta\to\gamma \qquad f:\gamma\to\delta}{f\circ g:\beta\to\delta}\,(?)}{(f\circ g)\circ h:\alpha\to\delta}\,(?)$$

Como justificamos?

ALVO: $(\forall a:\alpha)\,[\, \underbrace{((f\circ g)\circ h)\,a =_\delta (f\circ(g\circ h))\,a}_{\text{ALVO:}} \,]$

Seja $a:\alpha$. -- ALVO:

Calc: $((f\circ g)\circ h)\;a$

$= (f\circ g)\,(h\;a)$ $[\,(\circ).1 \quad g := f\circ g \quad f := h \quad a := a\,]$

$= f\,(g\,(h\;a))$ $[\,(\circ).1 \quad \ldots\,]$

$(f\circ(g\circ h))\;a$

$\overset{...}{=} f\,(g\,(h\;a))$

Associatividade sintáctica de ␣ : L

Associatividade sintáctica de $\rightarrow$: R

por quê?

Como justificamos?

$$\dfrac{f : \gamma \to \delta \qquad g : \beta \to \gamma}{\langle f, g \rangle : (\gamma \to \delta) \times (\beta \to \gamma)}\;(\times)\text{-}I$$

$$\dfrac{(\circ)_{\beta \to \gamma \to \delta} : ((\gamma \to \delta) \times (\beta \to \gamma)) \to (\beta \to \delta) \qquad \langle f, g \rangle : (\gamma \to \delta) \times (\beta \to \gamma)}{(\circ)_{\sqcup} \langle f, g \rangle \equiv f \circ g : \beta \to \delta}\;(\to)\text{-}E$$

$$\dfrac{f \circ g : \beta \to \delta \qquad h : \alpha \to \beta}{\langle f \circ g, h \rangle : (\beta \to \delta) \times (\alpha \to \beta)}\;(\times)\text{-}I$$

$$\dfrac{(\circ)_{\alpha \to \beta \to \delta} : ((\beta \to \delta) \times (\alpha \to \beta)) \to (\alpha \to \delta) \qquad \langle f \circ g, h \rangle : (\beta \to \delta) \times (\alpha \to \beta)}{(\circ)_{\sqcup} \langle f \circ g, h \rangle \equiv (f \circ g) \circ h : \alpha \to \delta}\;(\to)\text{-}E$$

Fun: $(\circ)$ & id

a identidade (do tipo α)

$$\mathrm{id}_\alpha : \alpha \to \alpha$$

$$\mathrm{id}_\alpha \overset{\text{def}}{\equiv} \lambda x : \alpha \,.\, x \qquad \textbf{ou} \qquad \mathrm{id}_\alpha\, x \overset{\text{def}}{\equiv} x$$

Θ. A id merece seu nome

$$(\circ)_{\alpha\to\alpha\to\beta} : (\alpha \to \beta) \times (\alpha \to \alpha) \to (\alpha \to \beta)$$

A id_α é uma identidade da $(\circ)_{\alpha\to\alpha\to\beta}$

$$\Updownarrow$$

$$f \circ \mathrm{id} = f \quad \& \quad \boxed{\mathrm{id} \circ f = f}$$

TYPE ERROR!

$\underset{: \mathrm{Int}}{1}$ é id de $\underset{: \mathrm{Int} \times \mathrm{Int} \to \mathrm{Int}}{(\cdot)}$

$$(\forall x : \mathrm{Int})\,[\, 1 \cdot x = x = x \cdot 1 \,]$$

θ. A id merece seu nome

$$f \circ_{?} id_\alpha \overset{?}{=} f \overset{?}{=} id_\beta \circ_{?} f$$

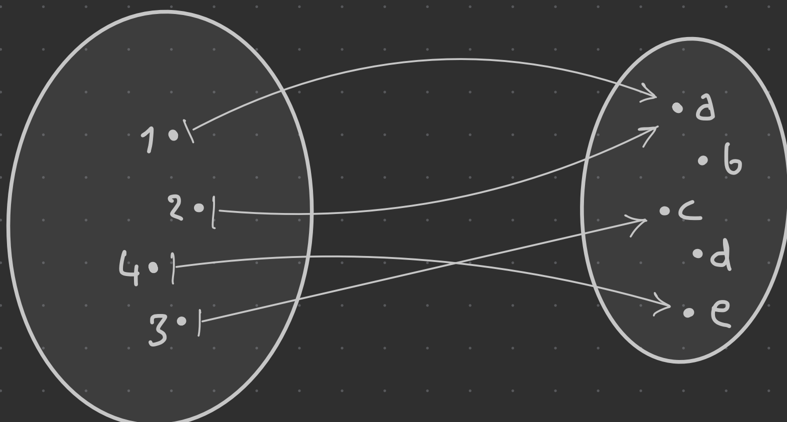
$$\begin{tikzcd}
\alpha \arrow[r, "id_\alpha"] \arrow[dr, bend right, "f"'] \arrow[dr, phantom, shift left=3, "="] & \alpha \arrow[d, "f"] \arrow[dr, bend left, "f"] \arrow[dr, phantom, shift left=3, "="] & \\
& \beta \arrow[r, "id_\beta"'] & \beta
\end{tikzcd}$$

Diagramas internos...

lec08

2025-04-11

$$A \xrightarrow{f} B$$



Funções:

- determinabilidade
- totalidade

...e externos

$$A \xrightarrow{f} B$$

-jetividades: inj, sobre, bij

surj

$\neg\neg P \overset{?}{\Rightarrow} P$

$\neg\neg P \overset{?}{\Leftarrow} P$

f respeita as $(=)$

$(\forall a, a' : A)[\ a = a' \Rightarrow fa = fa'\ ]$

$A \xrightarrow{f} B$

f respeita as distinções

f inj $\overset{\text{def}}{\iff} (\forall a, a' : A)[\ a \neq a' \Rightarrow fa \neq fa'\ ]$

$? \Uparrow \Downarrow ?$

$fa = fa' \Rightarrow a = a'$

f surj / sobre $\overset{\text{def}}{\iff} (\forall b : B)(\exists a : A)[\ a \overset{f}{\longmapsto} b\ ]$

$\Updownarrow$

$fa = b$

f bij $\overset{\text{def}}{\iff} f$ inj & f sobre

$\Updownarrow ?$

$(\forall b : B)(\exists ! a : A)[\ fa = b\ ]$ $(\forall a : A)(\exists ! b : B)[\ fa = b\ ]$

(apenas função !

Respeito & preservação

também: $(\circ)$ preserva inj, sobre, bij

Θ. A $(\circ)$ respeita: inj, sobre, bij

$$\Updownarrow$$

$$(\forall A \xrightarrow{f} B \xrightarrow{g} C)[\, f \text{ inj} \;\&\; g \text{ inj} \Rightarrow g \circ f \text{ inj} \,]$$

Abstraindo com buracos

$$(\ ) \ \langle (+) \ \langle 3, 42 \rangle, 2 \rangle$$

$$|||$$

$$(3 + \underbrace{42}_{:\,\text{Int}}) \cdot 2 : \text{Int}$$

$$(3 + \_) \cdot 2 : \text{Int} \to \text{Int}$$

$$(3 \overset{:\,\text{Int} \times \text{Int} \to \text{Int}}{+} 42) \cdot 2 : \text{Int}$$

$$(3 \ \_ \ 42) \cdot 2 : ((\text{Int} \times \text{Int}) \to \text{Int}) \to \text{Int}$$

$$|||$$

$$(\_ \ \langle 3, 42 \rangle) \cdot 2$$

$$(3 + \_) \cdot 2 \ : \ \mathrm{Int} \to \mathrm{Int}$$

$$(3 + \_) \cdot \_ \ : \ \mathrm{Int} \to (\mathrm{Int} \to \mathrm{Int})$$

?

$$(3 + \_) \cdot \_ \ : \ ? \ \begin{cases} \mathrm{Int} \to (\mathrm{Int} \to \mathrm{Int}) \\ \mathrm{Int} \times \mathrm{Int} \to \mathrm{Int} \end{cases}$$

$$\lambda x.\ \lambda y.\ (3 + y) \cdot x \ : \ \mathrm{Int} \to (\mathrm{Int} \to \mathrm{Int})$$

$$\lambda w.\ (3 + (w.r)) \cdot (w.l) \ : \ \mathrm{Int} \times \mathrm{Int} \to \mathrm{Int}$$

Isomorfismo

iso (igual)
homo (parecido)

$(\cong) \; : \; \mathrm{Type} \times \mathrm{Type} \to \mathrm{Prop}$

$\alpha \cong \beta \overset{\text{def}}{\iff} (\exists f : \alpha \to \beta)[\, f \text{ iso} \,]$

(α é isô̂morfo ao β

α, β são isô̂morfos ou isomórficos (entre si))

$\Theta!$ $(\cong)$ é simétrica

$\_ \text{ iso} \; : \; (\alpha \to \beta) \to \mathrm{Prop}$

$f : \alpha \to \beta \text{ iso} \overset{\text{def}}{\iff} (\exists f' : \alpha \leftarrow \beta)[\, f' \circ f = id_\alpha \;\&\; f \circ f' = id_\beta \,]$

$f : \alpha \xrightarrow{\cong} \beta \quad , \quad f : \alpha \cong \beta$

$\Theta.$ $(\cong)$ é uma relação de equivalência (refl; trans; sym)

Currificação

currificada (binária)

$$\begin{tikzcd} (\alpha \times \beta) \to \gamma \arrow[r, bend left, "\mathrm{curry}"] \arrow[r, phantom, "\overset{?}{\cong}"] & \alpha \to (\beta \to \gamma) \arrow[l, bend left, "\mathrm{uncurry}"] \end{tikzcd}$$

descurrificada

HW

curry $\ldots \overset{\mathrm{def}}{\equiv} \ldots$

uncurry $\ldots \overset{\mathrm{def}}{\equiv} \ldots$

Aritmética de tipos

Qual é o seu papel nesta conversa?

$(\to)$

$(\times)$

$\mathbb{0}$	$\alpha \times \mathbb{0} \overset{?}{\cong} \mathbb{0}$	$\mathbb{0}$ é um $(\times)$-absorvente-R (anulador) $\mathbb{0}$ absorve a $(\times)$ pela R
$\mathbb{1}$	$\alpha \times \mathbb{1} \cong \alpha$	$\mathbb{1}$ é uma $(\times)$-identidade-R
	$\alpha \times \beta \cong \beta \times \alpha$	$(\times)$ é comutativa
	$(\alpha \times \beta) \times \gamma \cong \alpha \times (\beta \times \gamma)$	$(\times)$ é associativa

$(+)$? $\mathbb{2}$?

HW

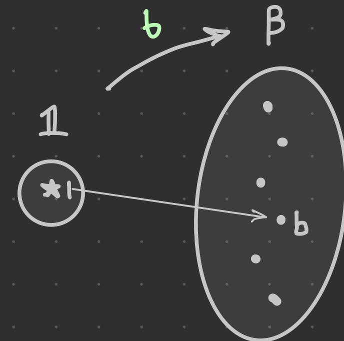
Funções saíndo/entrando

lec09

2025-04-14

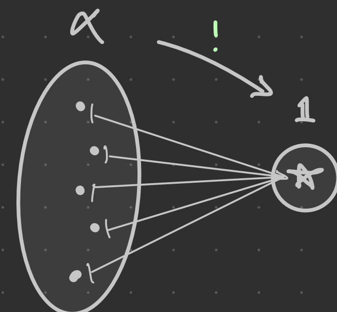
$\mathbb{1}$

saindo: $\mathbb{1} \xrightarrow{b} \beta$



$b : \beta \quad \rightsquigarrow \quad \star \mapsto b$

$\mathbb{1} \to \beta \overset{?}{\cong} \beta$

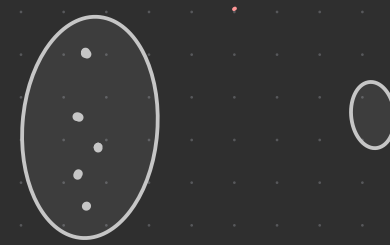


entrando: $\alpha \xrightarrow{!} \mathbb{1}$

$\varnothing$

saindo: $\varnothing \xrightarrow{!} \beta$ $\qquad$ $\varnothing \xrightarrow{!} \beta$

entrando: $a \to \varnothing$ $\qquad$ $\alpha \xrightarrow[?]{} \varnothing$



(se $\alpha \cong \varnothing$

então temos !

senão, não!

Construções e cardinalidades

$$\alpha + \beta \cong \beta + \alpha$$

$$|\alpha \times \beta| = |\alpha| \cdot |\beta| \qquad |A \times B| = |A| \cdot |B|$$

yikes!

$$|\alpha + \beta| = |\alpha| + |\beta| \qquad |A \cup B| \leq |A| + |B|$$

Funções de graça (construções)

$$
\begin{tikzcd}
\mathbb{1} \arrow[loop left, "1_{\mathbb{1}}"] & \alpha \arrow[l, "!"'] \arrow[d, "f"] \arrow[dr, bend left, "g\circ f"] \arrow[loop above, "1_\alpha"] \arrow[l, phantom] & \\
& \beta \arrow[r, "g"'] \arrow[ul, bend left, "!"] \arrow[loop below, "1_\beta"] & \gamma \arrow[ull, bend right, "!"'] \arrow[loop below, "1_\gamma"]
\end{tikzcd}
$$

$$! \circ f = !$$

Produto de funções $(f \times g)$

$$
\begin{tikzcd}
\alpha \arrow[d, "f"'] & \alpha \times \beta \arrow[l, "outl_{\alpha\times\beta}"'] \arrow[r, "outr_{\alpha\times\beta}"] \arrow[d, dotted, "?\ f\times g"] & \beta \arrow[d, "g"] \\
\gamma & \gamma \times \delta \arrow[l, "outl_{\gamma\times\delta}"] \arrow[r, "outr_{\gamma\times\delta}"'] & \delta
\end{tikzcd}
$$

$$f \circ outl_{\alpha\times\beta} = outl_{\gamma\times\delta} \circ (f \times g)$$

$$g \circ outr_{\alpha\times\beta} = outr_{\gamma\times\delta} \circ (f \times g)$$

$$f \times g \ :\ \alpha \times \beta \longrightarrow \gamma \times \delta$$

$$(f \times g)\ w \ \stackrel{def}{\equiv}\ \langle f\ (w.l),\ g\ (w.r) \rangle$$

$$(f \times g)\ \langle a, b \rangle \ \stackrel{def}{\equiv}\ \langle f\ a,\ g\ b \rangle$$

Tentativa : definir $f \cup g$

$$\begin{tikzcd}
\alpha \arrow[r, "\lambda x.x", "\subseteq"'] \arrow[d, "f"'] \arrow[dr, phantom, "="] & \alpha \cup \beta \arrow[d, dotted, "?\ f \cup g"] & \beta \arrow[l, "\lambda x.x"', "\supseteq"] \arrow[d, "g"] \\
\gamma \arrow[r, "\subseteq"'] & \gamma \cup \delta & \delta \arrow[l, "\supseteq"]
\end{tikzcd}$$

$i : \alpha \to \alpha \cup \beta$

$i\ 1 = 1$ $\quad \lambda x.x :$

$i\ 2 = 2$

$3 \quad 3$

$4 \quad 4$

$$(f \cup g) : \alpha \cup \beta \to \gamma \cup \delta$$

$$(f \cup g)\ u \overset{\text{def}}{\equiv} \begin{cases} f\ u, & \text{caso } u \in \alpha; \\ g\ u, & \text{caso } u \in \beta \setminus \alpha. \end{cases}$$

Soma

$$\frac{\alpha \ \ \text{type} \qquad \beta \ \ \text{type}}{\alpha+\beta \ \ \text{type}}\ (+)\text{-F}$$

$$\frac{a : \alpha}{l \cdot a : \alpha+\beta}\ (+)\text{-I}_1 \qquad\qquad \frac{b : \beta}{r \cdot b : \alpha+\beta}\ (+)\text{-I}$$

inl a $\quad \iota\, a$ $\qquad\qquad$ inr b $\quad \iota\, b$

$$\frac{s : \alpha+\beta \qquad f : \alpha \to \gamma \qquad g : \beta \to \gamma}{\text{case } s \text{ of } \left\{ \begin{matrix} l \cdot a \rightsquigarrow f\, a \\ r \cdot b \rightsquigarrow g\, b \end{matrix} \right] : \gamma}\ (+)\text{-E}$$

$$s \left\{ \begin{matrix} f \\ g \end{matrix} \right] \qquad \left\{ \begin{matrix} f \\ g \end{matrix} \right] s \qquad \left\{ \begin{matrix} f \\ g \end{matrix} \right] : \alpha+\beta \to \gamma$$

Computação (β):

$$\text{case } \ell \cdot x \text{ of } \begin{Bmatrix} \ell \; a \to f \; a \\ r \cdot a \to g \; a \end{Bmatrix} \rightsquigarrow f \; x$$

$$\text{case } r \cdot x \text{ of } \begin{Bmatrix} \ell \; a \to f \; a \\ r \cdot a \to g \; a \end{Bmatrix} \rightsquigarrow g \; x$$

ou

$$\overset{\beta}{=}$$

$$(\ell \; x) \begin{Bmatrix} f \\ g \end{Bmatrix} \rightsquigarrow f \; x$$

$$(r \cdot x) \begin{Bmatrix} f \\ g \end{Bmatrix} \rightsquigarrow g \; x$$

(η)?

Aritmética de tipos

$$(\alpha + \beta) + \gamma \cong \alpha + (\beta + \gamma)$$

$$\alpha + \beta \cong \beta + \alpha$$

$$\left.\begin{aligned} 0 + \alpha &\cong \alpha \\ \delta \times (\alpha + \beta) &\cong (\delta \times \alpha) + (\delta \times \beta) \end{aligned}\right\} \text{distributividades}$$

$$1 + 1 \cong 2 \cong \mathbb{B}$$

Cadê a exponenciação?

$$\alpha^{\beta + \gamma} \cong \alpha^{\beta} \times \alpha^{\gamma} \qquad \alpha^{\beta \cdot \gamma} \cong (\alpha^{\beta})^{\gamma}$$

União disjuntora (ou "disjunta")

lec10

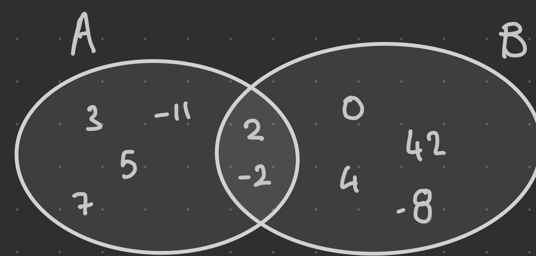
2025-04-25

$\alpha + \beta$ Como fazer isso num mundo un(i)typed?

$\ell \cdot a_1$

$\ell \cdot a_2$

$r \cdot b_1$

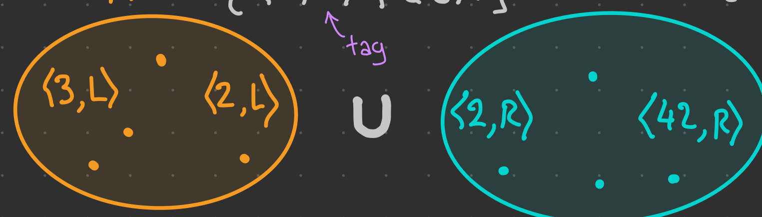


$\mathbf{A} \overset{\text{def}}{\equiv} \{ \langle a, L \rangle \mid a \in A \}$ tag

$\mathbf{B} \overset{\text{def}}{\equiv} \{ \langle b, R \rangle \mid b \in B \}$ tag

união-disjunta

$A \uplus B \overset{\text{def}}{\equiv}$



$\ell \cdot \_ : A \longrightarrow A \uplus B$

$\ell \cdot a \overset{\text{imp}}{\equiv} \langle a, L \rangle$

$\ell \cdot \_ \overset{\text{imp}}{\equiv} \langle \_, L \rangle$

$r \cdot \_ : \ldots$

$\mathbf{A} \overset{\text{def}}{\equiv} A \times \{L\}$

$\mathbf{B} \overset{\text{def}}{\equiv} B \times \{R\}$

$A \times B \overset{\text{def}}{\equiv} \{ \langle a, b \rangle \mid a \in A,\ b \in B \}$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\alpha & \alpha \times \beta \arrow[l, "\pi_1 \; \mathrm{outl}"', "\_\cdot l"] \arrow[r, "\pi_2 \; \mathrm{outr}", "\_\, r"'] & \beta \\
& \gamma \arrow[ul, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[u, dotted, "h?\ \exists!"] &
\end{tikzcd}
$$

HW

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\alpha \arrow[r, "\iota_1 \; \mathrm{inl}", "l\cdot\_"'] \arrow[dr, bend right, "f"'] & \alpha + \beta \arrow[d, dotted, "h?\ \exists!"] & \beta \arrow[l, "\iota_2 \; \mathrm{inr}"', "r\cdot\_"] \arrow[dl, bend left, "g"] \\
& \gamma &
\end{tikzcd}
$$

$h\ c \stackrel{def}{\equiv} \langle f\ c\ ,\ g\ c \rangle$

$(h\ c)\cdot l \stackrel{def}{\equiv} f\ c$

$(h\ c)\cdot r \stackrel{def}{\equiv} g\ c$

$f = \mathrm{outl} \circ h$ pairing

$g = \mathrm{outr} \circ h$ $\langle f, g \rangle$

pair(f, g)

como entrar num produto

$h\ (l\cdot a) \stackrel{def}{\equiv} f\ a$

$h\ (r\cdot b) \stackrel{def}{\equiv} g\ b$

$h\ s \stackrel{def}{\equiv} \left(\text{case } s \text{ of } \left\{ \begin{array}{l} l\cdot a \rightsquigarrow f\ a \\ r\cdot b \rightsquigarrow g\ b \end{array} \right] \right)$

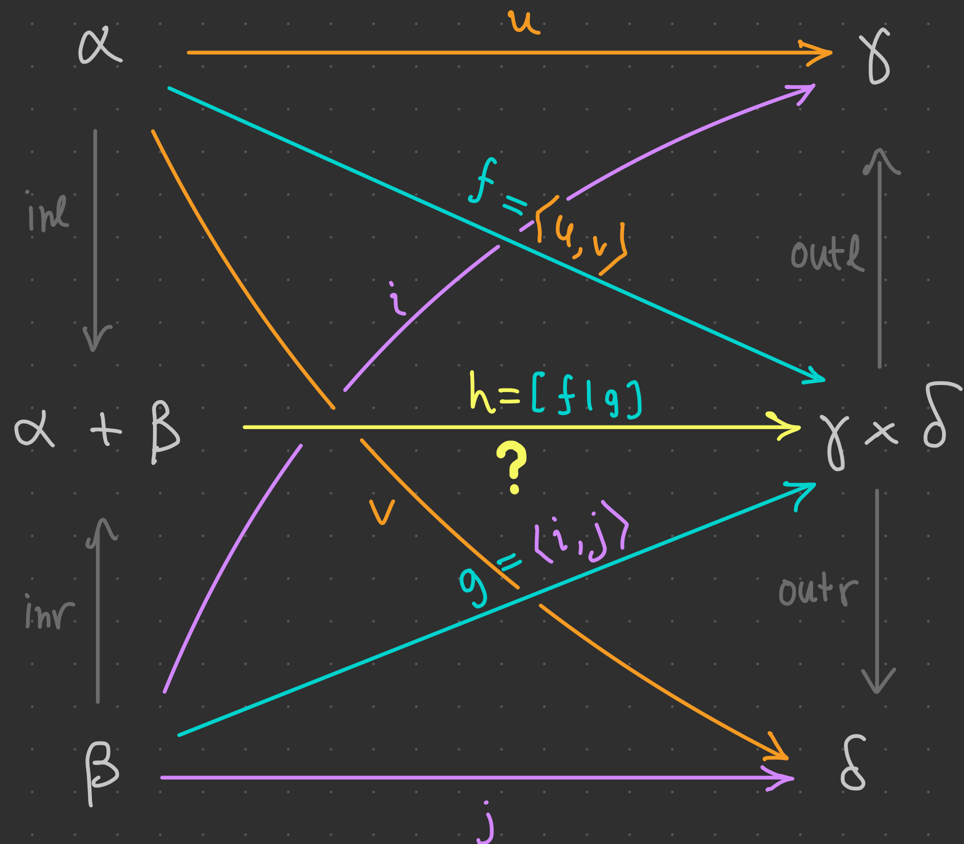
$f = h \circ \mathrm{inl}$

$g = h \circ \mathrm{inr}$

copairing

copair(f, g) $[f \mid g]$ $\left\{ \begin{array}{l} f \\ g \end{array} \right]$

como sair duma soma



$$[\langle u,v\rangle \mid \langle i,j\rangle] \stackrel{?}{=} \langle\ ?\ \rangle$$

Point-free (pointless, tácito)

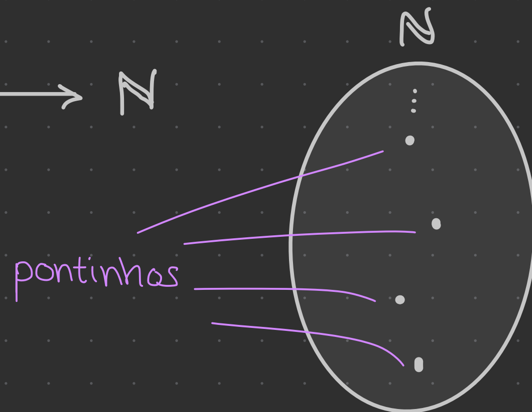
$$
\begin{tikzcd}
\alpha \arrow[rrr, "?"] \arrow[dr, "?"'] \arrow[drr, bend right, "?"'] & & & \omega \\
& \beta \arrow[r, "?"'] \arrow[urr, "?"] & \gamma \arrow[ur, "?"'] &
\end{tikzcd}
$$

Devemos pensar nomes!

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$

$$f\ n \stackrel{\text{def}}{\equiv} (2 \cdot n + 1)^2$$

pontinhos



$$\begin{tikzcd}
\mathbb{N} \arrow[rrr, bend left, "f"] \arrow[dr, "double"'] & & & \mathbb{N} \\
& \mathbb{N} \arrow[r, "succ"'] \arrow[u, phantom, "="] & \mathbb{N} \arrow[ur, "square"'] &
\end{tikzcd}$$

$$f \stackrel{\text{def}}{\equiv} square \circ succ \circ double$$

ou: $double \ ;\ succ \ ;\ square$

...E o double?

$$\begin{tikzcd}
\mathbb{N} \arrow[rr, "double"] \arrow[dr, bend right, "\Delta \\ diag"'] & & \mathbb{N} \\
& \mathbb{N} \times \mathbb{N} \arrow[ur, bend right, "(+)"'] &
\end{tikzcd}$$

$$\left.\begin{aligned}
\Delta &: \alpha \to \alpha \times \alpha \\
\Delta\ x &\stackrel{\text{def}}{\equiv} \langle x, x \rangle
\end{aligned}\right\}$$

Dá para definir a Δ (diag) sem pontos?

$$f \stackrel{\text{def}}{\equiv} \underbrace{(\times) \circ \Delta}_{\text{square}} \circ \text{succ} \circ \underbrace{(+) \circ \Delta}_{\text{double}}$$

Mais construções de funções (fun de graça)

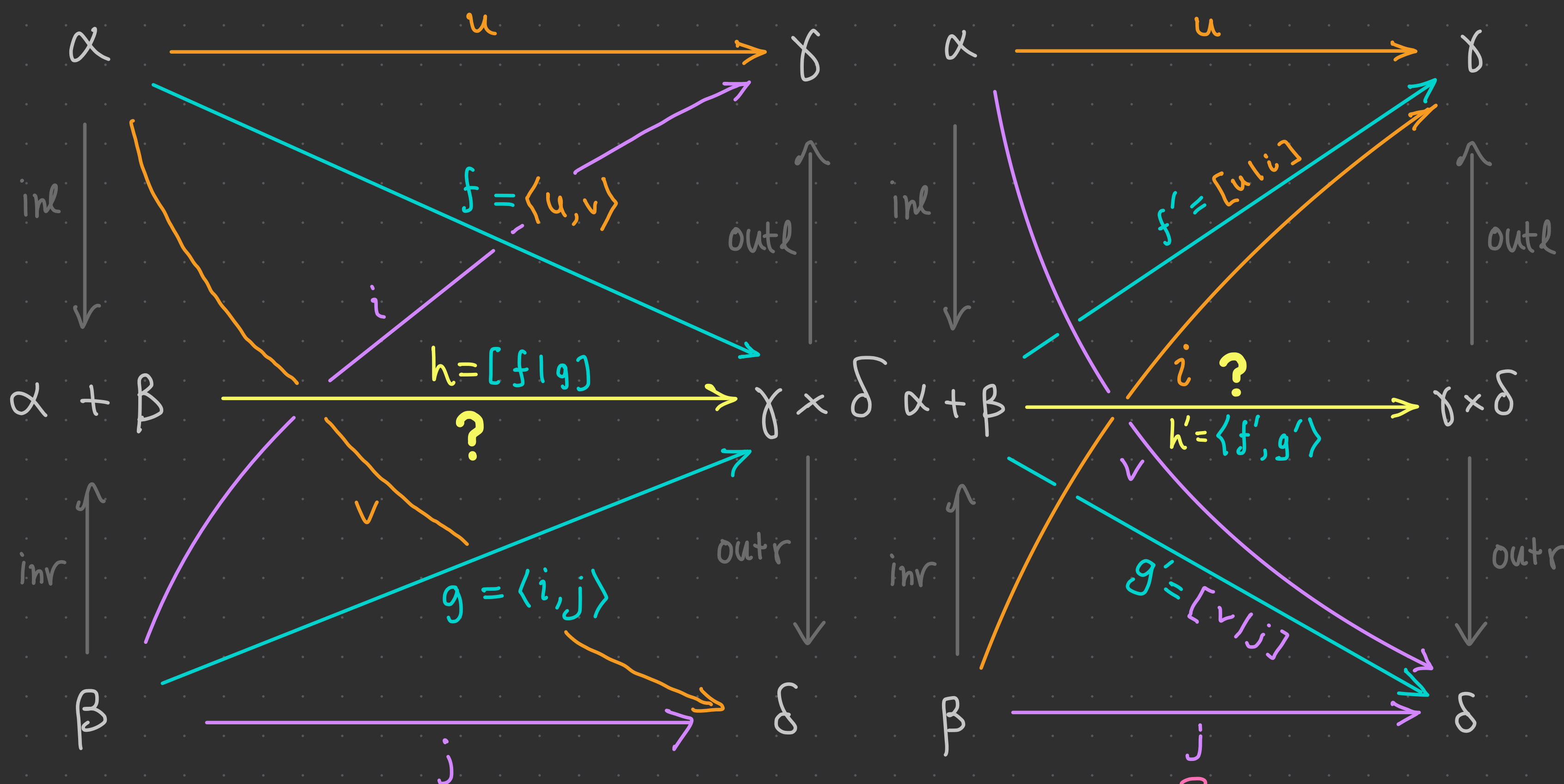
Point-wise

$$(*) : A \times A \longrightarrow A$$
$$f, g : D \longrightarrow A$$

$$(*) : (\delta \to \alpha) \times (\delta \to \alpha) \longrightarrow (\delta \to \alpha)$$
$$(f * g)\ d \overset{\text{def}}{\equiv} (f\ d) * (g\ d)$$

... e se $D \xrightarrow{f} A \xleftarrow{g} D'$?

$$\text{age} : \text{Person} \longrightarrow \mathbb{N}$$
$$\text{children} : \text{Person} \longrightarrow \mathbb{N}$$
$$(\cdot) : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$
$$(\text{age} \cdot \text{children}) : \text{Person} \longrightarrow \mathbb{N}$$

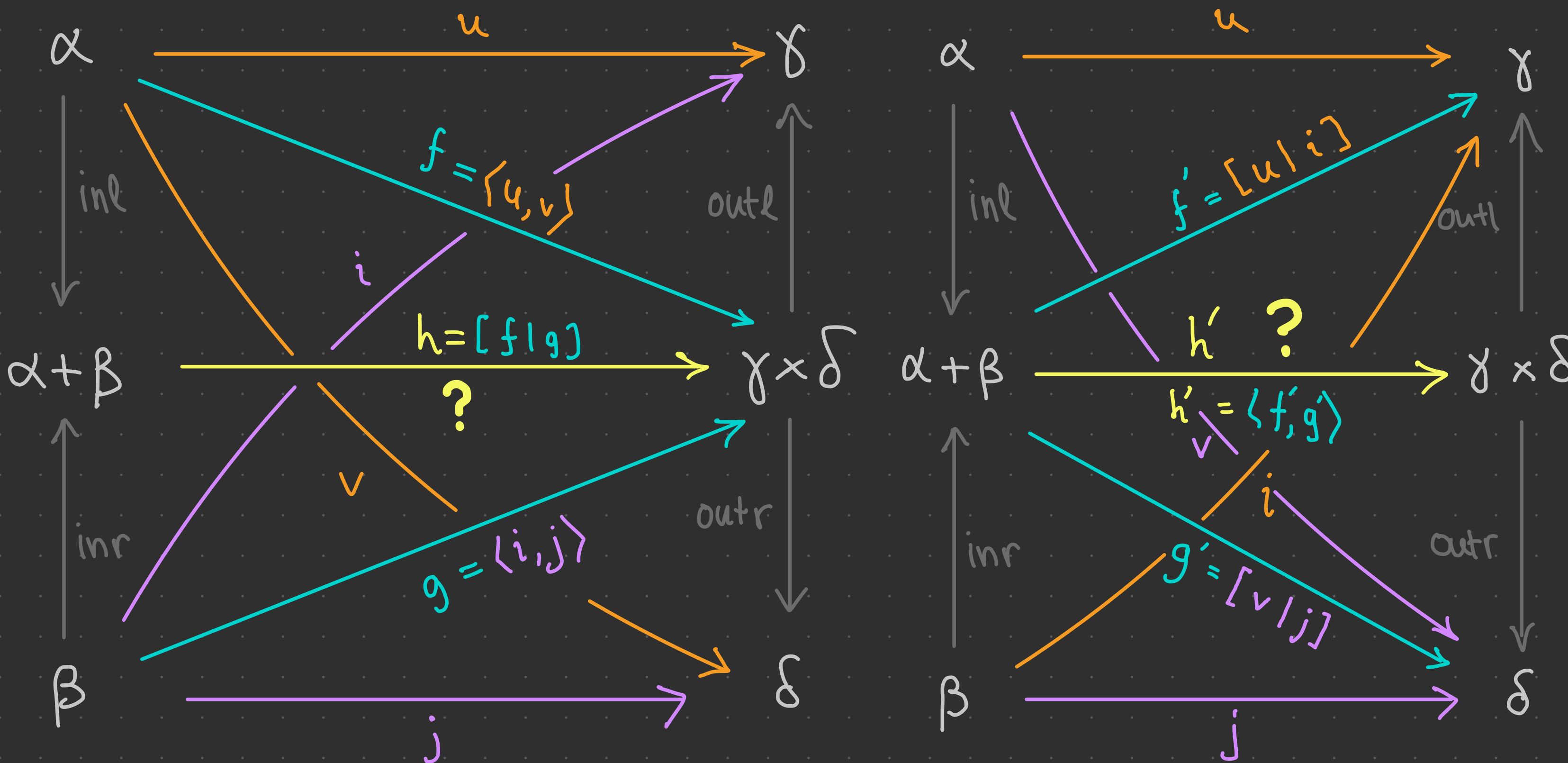


$$[\langle u,v\rangle \mid \langle i,j\rangle] \stackrel{?}{=} \langle \quad ? \quad \rangle$$

lec11

2025-04-28

$$\overset{h}{[\langle u,v\rangle \mid \langle i,j\rangle]} \overset{?}{=} \overset{h'}{\langle [u \mid i]\ [v \mid j]\rangle}$$



Mais construções de fun

identidade

$$\mathrm{id}_\alpha : \alpha \to \alpha$$

$$\mathrm{id}_\alpha \overset{\text{def}}{\equiv} \lambda x \,.\, x$$

constante

$$k_{c:\alpha} : \delta \to \alpha$$

$$k_{c:\alpha} \overset{\text{def}}{\equiv} \lambda x \,.\, c$$

$$k : \alpha \to (\delta \to \alpha)$$

$$k\, c \overset{\text{def}}{\equiv} \lambda x \,.\, c$$

iterações

endomapa

$$f : \alpha \to \alpha$$

$$f^{\circ n} : \alpha \to \alpha$$

$$f^{\circ 0} \overset{\text{def}}{\equiv} \mathrm{id}_\alpha$$

$$f^{\circ Sn} \overset{\text{def}}{\equiv} f \circ f^{n}$$

$$f^{\circ -}_{\alpha} : \mathbb{N} \to (\alpha \to \alpha)$$

$$f : \alpha \to \beta$$

$$f \text{ constante} \overset{\text{def}}{\iff} (\exists c : \beta)[\, f = k_c \,]$$

?

$$f \text{ invariável} \overset{\text{def}}{\iff} (\forall a, a' : \alpha)[\, f\,a = f\,a' \,]$$

lec12

2025-04-30

inversa

$f : \alpha \to \beta$

$$\underbrace{f' \circ_{\alpha\to\beta\to\alpha} f =_{\alpha\to\alpha} \mathrm{id}_\alpha}_{f' \text{ é } (\circ)\text{-invL de } f} \quad \& \quad \underbrace{f \circ_{\beta\to\alpha\to\beta} f' =_{\beta\to\beta} \mathrm{id}_\beta}_{f' \text{ é } (\circ)\text{-invR de } f}$$

f invertível $\overset{\text{def}}{\iff} (\exists f' : \alpha \leftarrow \beta)[\overbrace{f' \text{ é } (\circ)\text{-inv } f}]$

$\Updownarrow$ **HW**

f bijetiva $\overset{\text{def}}{\iff} f$ inj $\&$ f sobrej

$f^{-1} \overset{\text{def}}{\equiv}$ a $(\circ)$-inv de f

$f^{-1}\, y = x \overset{\text{def}}{\iff} y = f\, x$

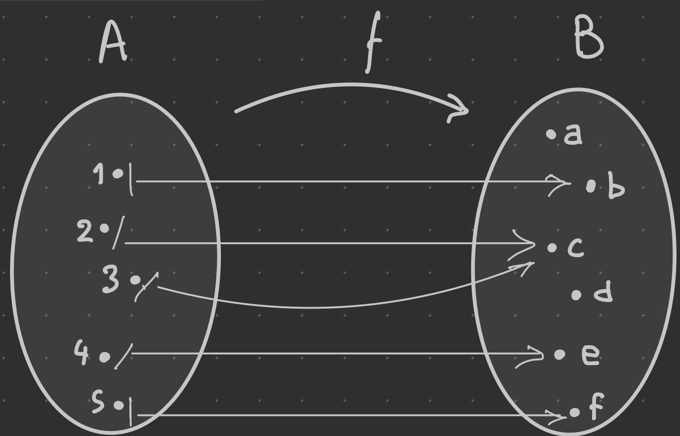
$x \xleftarrow{f^{-1}} y \overset{\text{def}}{\iff} x \overset{f}{\longmapsto} y$

$\_^{-1}$ respeita a bijetividade?

$(f^{-1})^{-1} = ?$

HW

imagem



$$\vec{\wp}A \xrightarrow{\vec{\wp}f} \vec{\wp}B$$

$$\{4,5\} \longmapsto \{e,f\}$$

$$\{2,3\} \longmapsto \{c\}$$

$$\{3,4\} \longmapsto \{c,e\}$$

$\wp f \; A \; = \;$ Range(f)
Image(f)

$$\wp f \; \emptyset_A = \emptyset_B$$

$$\wp f : \wp A \to \wp B$$

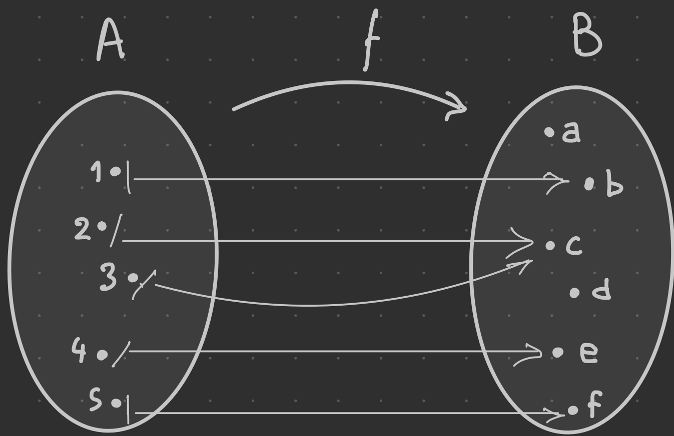
$$\wp f \; X \overset{\text{def}}{\equiv} \{ f x \mid x \in X \}$$

$$y \in \wp f \; X \overset{\text{def}}{\iff} (\exists x \in X)[x \overset{f}{\longmapsto} y]$$

a f-imagem de X

a imagem de X através da f

préimagem



$$\overleftarrow{\wp}A \xleftarrow{\overleftarrow{\wp}f} \overleftarrow{\wp}B$$

$$\{4,5\} \longleftarrow\!\shortmid \{e,f\}$$

$$\{1,2,3\} \longleftarrow\!\shortmid \{b,c\}$$

$\mathrm{Set}\ \alpha \longleftarrow \mathrm{Set}\ \beta$

$$\overleftarrow{\wp}f \ : \ \wp A \longleftarrow \wp B$$

ou: $\overleftarrow{\wp}f\ Y \overset{\text{def}}{=} \{ x \mid \ldots \}$

$$x \in \boxed{\overleftarrow{\wp}f\ Y} \overset{\text{def}}{\iff} f\,x \in Y$$

a f-préimagem de Y

a préimagem de Y através da f

Outras notações de (pré)imagens

$$\left.\begin{array}{l} f[X] \\ f(X) \\ f\sqcup X \end{array}\right\} \equiv \overrightarrow{\wp} f\ X \qquad\qquad \left.\begin{array}{l} f^{-1}[Y] \\ f^{-1}(Y) \\ f^{-1}\sqcup Y \end{array}\right\} \equiv \overleftarrow{\wp} f\ Y$$

$$f^{-1}[Y] \equiv \left\langle \begin{array}{c} \overrightarrow{\wp}(f^{-1})\ Y \\ \overset{?}{\|} \\ \overleftarrow{\wp} f\ Y \end{array} \right.$$

HW:

Justifique a notação: demonstre que quando há possibilidade de ambigüidade na intensão (i.e quando f invertível) temos $\overrightarrow{\wp}(f^{-1})\ Y = \overleftarrow{\wp} f\ Y$.

Propriedades (HW)

$$\vec{\wp} f\,(X \cup Y) \overset{?}{=} (\vec{\wp} f\ X) \cup (\vec{\wp} f\ Y)$$

$$\cap \qquad\qquad \cap$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$$\begin{tikzcd} \vec{\wp} A \arrow[r, "\vec{\wp} f"] \arrow[rr, bend right, "\vec{\wp}(g \circ f)"'] & \vec{\wp} B \arrow[r, "\vec{\wp} g"] & \vec{\wp} C \end{tikzcd}$$

$$\vec{\wp}(g \circ f) \overset{?}{=} \vec{\wp} g \circ \vec{\wp} f$$

$$A \xrightarrow{id_A} A$$

$$\begin{tikzcd} \wp A \arrow[r, "\wp\, id_A"] \arrow[r, bend right, "id_{\wp A}"'] \arrow[r, phantom, "\overset{?}{=}"] & \wp A \end{tikzcd}$$

$$\vec{\wp}(1_A) : \vec{\wp} A \to \vec{\wp} A$$

$$\overset{?}{=}$$

$$1_{\vec{\wp} A}$$

$\overleftarrow{\wp}$?

$$\alpha \xrightarrow{f} \beta$$

$\Big\downarrow$ Set — aplicável tanto nos Objetos de interesse quanto nas setas de interesse

$$\text{Set } \alpha \xrightarrow{\text{Set } f} \text{Set } \beta$$

Famílias indexadas (I-tuplas)

P : Person

Seja F_p o conjunto dos filhos de p.

$(F_p \mid p \in P)$ $\quad (F_p)_{p \in P}$ $\quad$ P-indexada

$$\bigcup_{p \in P} F_p$$

$(F_i \mid i \in I)$ $\quad (F_i)_{i \in I}$ $\quad$ I-indexada

Seqüências

lec13

2025-05-09

Temos definido para cada $n : \mathbb{N}$ um objeto $a_n : \alpha$.

Assim temos uma seqüência de α ($\mathrm{Seq}\ \alpha$).

$$n : \mathbb{N} \vdash a_n : \alpha$$

Notações Seq-builder:

ωμέγα

$$(a_n)_{n=0}^{\infty} \quad (a_n)_{n\geq 0} \quad (a_n)_{n\in\mathbb{N}} \quad (a_n)_{n\in\omega} \quad (a_n \mid n \in \omega) \quad (a_n)_n : \mathrm{Seq}\ \alpha$$

$$(a_n)_{n=0}^{\infty} \equiv (a_0, a_1, a_2, \ldots)$$

$$(a_{2n})_{n=7}^{\infty} \equiv (a_7, a_{14}, a_{28}, \ldots)$$

$$\{a_n\}_n : \mathrm{set}\ \alpha \quad \text{vs} \quad (a_n)_n : \mathrm{seq}\ \alpha$$

a ordem usual do $\mathbb{N}$

Conjunto parcialmente ordenado

$$\omega \overset{\text{def}}{\equiv} (\mathbb{N} ; \leq) : \text{Poset ?}$$

?

Seqüências de conjuntos $\bigcup_\alpha : \text{Set}(\text{Set}\ \alpha) \to \text{Set}\ \alpha$

$A : \text{Seq}(\text{Set}\ \alpha) \qquad A \equiv (A_0, A_1, \ldots) \equiv (A_n)_n$

$\bigcup_{n=0}^{\infty} A_n \equiv\ ?$

$\bigcup_{n=0}^{\infty} A_n \equiv A_0 \cup A_1 \cup A_2 \cup \cdots$

$\heartsuit : \alpha \times \alpha \to \alpha$

de binária para n-aria : assoc + id

$\heartsuit : \text{FinList}\ \alpha \to \alpha$

Não é gratuito a partir da $-\cup-$ + assoc + id.

Mas é a partir da coerção $\text{Seq}\ \alpha \rightsquigarrow \text{Set}\ \alpha$. $\bigcup_{n=0}^{\infty} A_n \equiv \bigcup \{A_n\}_n$

Equivalentemente:

$\bigcup (A_n)_{n=s}^{\infty}$

$: \text{Seq}(\text{Set}\ \alpha) \to \text{Set}\ \alpha$

$x \in \bigcup_{n=s}^{\infty} A_n \overset{\text{def}}{\iff} (\exists n \geq s)[x \in A_n] \overset{\heartsuit}{\iff}$ x pertence àlgum dos A's

$x \in \bigcap_{n=s}^{\infty} A_n \overset{\text{def}}{\iff} (\forall n \geq s)[x \in A_n] \overset{\heartsuit}{\iff}$ x pertence a todos os A's

Sequência de uniões finitas de conjuntos: $\left(\bigcup_{n=0}^{t} A_n\right)_{t=0}^{\infty}$

$$
\mathrm{Seq}\ (\mathrm{Set}\ \alpha)
\left\{
\begin{array}{lll}
A_0 & : \mathrm{Set}\ \alpha & \equiv \bigcup_{n=0}^{0} A_n \\
A_0 \cup A_1 & : \mathrm{Set}\ \alpha & \equiv \bigcup_{n=0}^{1} A_n \\
A_0 \cup A_1 \cup A_2 & & \equiv \bigcup_{n=0}^{2} A_n \\
A_0 \cup A_1 \cup A_2 \cup A_3 & & \\
\vdots & &
\end{array}
\right.
$$

Desafio:

$$\bigcup_{n=0}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

$\cup|$? $\cap|$

$$\bigcap_{n=0}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

$$\equiv \underbrace{\bigcap_{m=0}^{\infty} A_m}_{\text{algo}(0)} \cup \underbrace{\bigcap_{m=1}^{\infty} A_m}_{\text{algo}(1)} \cup \cdots$$

$$\equiv (A_0 \cap A_1 \cap \cdots) \underset{\subseteq}{\cup} (A_1 \cap A_2 \cap \cdots) \underset{\subseteq}{\cup} \cdots$$

é uma $(\subseteq)$-chain

$$\equiv (A_0 \cup A_1 \cup \cdots) \underset{\supseteq}{\cap} (A_1 \cup A_2 \cup \cdots) \underset{\supseteq}{\cap} \cdots$$

Dicas

$$\bigcup \begin{cases} A_0 \cap A_1 \cap A_2 \cap A_3 \cap A_4 \cap \cdots \equiv \bigcap_{m=0}^{\infty} A_m \\ A_1 \cap A_2 \cap A_3 \cap A_4 \cap \cdots \equiv \bigcap_{m=1}^{\infty} A_m \\ A_2 \cap A_3 \cap A_4 \cap \cdots \equiv \bigcap_{m=2}^{\infty} A_m \\ \vdots \end{cases}$$

Exemplo: $\bigcap_{n=0}^{\infty} \{k\}_{k \geq n} = \emptyset$

$$x \in \bigcap_{n=0}^{\infty} \overbrace{\{n, n+1, n+2, \ldots\}}^{A_n}$$

$$\Updownarrow$$

$$(\forall n \geq 0)[\ x \in A_n\ ]$$

Mas $x \notin A_{x+1}$.

Definições recursivas de funções

Seja $a : \mathbb{N} \to \mathbb{N}$ definida pelas:

$$a\ 0 = 1$$

$$a\ n = n \cdot a\,(n-1)\ , \quad \text{para todo } n \geq 1$$

$$b\ 0 = 0$$

$$b\ 1 = 1$$

$$b\ n = b\,(n-1) + b\,(n-2)\ , \quad \text{para todo } n \geq 2$$

$$ev, od : \mathbb{N} \to \mathbb{N}$$

$$ev\ 0 = 1 \qquad od\ 0 = 0$$

$$ev\ (S\,n) = od\ n \qquad od\ (S\,n) = ev\ n$$

$$p\ x = \begin{cases} \log_2 x, & \text{caso } x \text{ pot de } 2 \\ p\ (x+1), & \text{c.c.} \end{cases}$$

$$f\ n = f\ n$$

$$g\ 0 = 1$$

$$g\ n = (g\ n) + 1$$

$$h\ n = h\ (n+1)$$

$$k\ 0 = 1$$

$$k\ 1 = 1$$

$$k\ n = \begin{cases} k(n/2), & \text{caso } n \text{ pot de } 2 \\ k(n+1), & \text{c.c.} \end{cases}$$

$$d\ 0 = 1$$

$$d\ 1 = 1$$

$$d\ n = \begin{cases} d(n/2), & \text{caso } n \text{ pot } 2 \\ d\ (n+3), & \text{c.c.} \end{cases}$$

$$c\ 0 = 1$$

$$c\ 1 = 1$$

$$c\ n = \begin{cases} c\ (n/2), & \text{caso } n \text{ par} \\ c\ (3n+1), & \text{c.c.} \end{cases}$$

Flashback do ensino fundamental

$$\begin{cases} 3x + 2y = 7 \\ x + 1 = 8 \end{cases} \quad ? : \mathbb{R} \times \mathbb{R}$$

Definições recursivas de funções

lec14
2025-05-12

Seja $a : \mathbb{N} \to \mathbb{N}$ definida pelas:

$$a\ 0 = 1$$
$$a\ n = n \cdot a\,(n-1)$$

determinado
resolução única (?)

$\cancel{fact : \mathbb{N} \to \mathbb{N}}$
$\cancel{fact\ 0 = 1}$
$\cancel{fact\ n = n \cdot fact\,(n-1)}$

$$f\ n = f\ n$$

trivial: qualquer $f : \mathbb{N} \to \mathbb{N}$ satisfaz. (refl)
(indeterminadíssimo)

$$g\ 0 = 1$$
$$g\ n = (g\ n) + 1$$

impossível: nenhuma $g : \mathbb{N} \to \mathbb{N}$ satisfaz.
$(g\ 1 \neq \mathrm{succ}\,(g\ 1))$ (FMC1: IRI)

$$h\ n = h\,(n+1)$$

Qualquer constante satisfaz

HW

Qualquer resolução é constante

indicadora ou característica

$C : \text{Set } \alpha$

$\chi_C : \alpha \to \mathbb{B}$ — ou algum conjunto/tipo numérico para abusar seus 0, 1

$$\chi_C\, a \stackrel{\text{def}}{\equiv} \begin{cases} 1 \ (\text{ou } 0), & \text{caso } a \in C \\ 0 \ (\text{ou } 1), & \text{c.c} \end{cases}$$

extenção

$f : A \to B$

$f' : A' \to B$

$A \subseteq A'$

f' extenção da $f \stackrel{\text{def}}{\iff}$

f, f' concordam no A

restrição

$f : A \to B$

$S \subseteq A$

$f \restriction S : S \to B$

$$\boxed{(f \restriction S)\, s \stackrel{\text{def}}{\equiv} f\, s}$$

ou: $f|_S$

co-restrição

$f : A \to B$

$S \supseteq B$

$f \downharpoonright S : A \to S$

$$\boxed{(f \downharpoonright S)\, a \stackrel{\text{def}}{\equiv} f\, a}$$

ou: $f|^S$

HW

Tem type error aqui...

Dá para resolver?

Generalização de produto

Elementos de conjuntos

$$
\begin{tikzcd}
\langle a, b \rangle \arrow[r, bend left, "\in"] \arrow[d, squiggly] & A \times B \arrow[d, squiggly] \\
(a_0, a_1) \arrow[r, bend left] \arrow[d, squiggly] & A_0 \times A_1 \arrow[d, squiggly] \\
(a_i \mid i \in \{0,1\}) \arrow[r, bend left] \arrow[d, squiggly] & \{ f : \{0,1\} \to A_0 \cup A_1 \mid f0 \in A_0 \ \& \ f1 \in A_1 \} \arrow[d, squiggly] \\
(a_i \mid i \in I) \arrow[r, bend left] & \{ f : I \to \bigcup_{i \in I} A_i \mid (\forall i \in I)[fi \in A_i] \}
\end{tikzcd}
$$

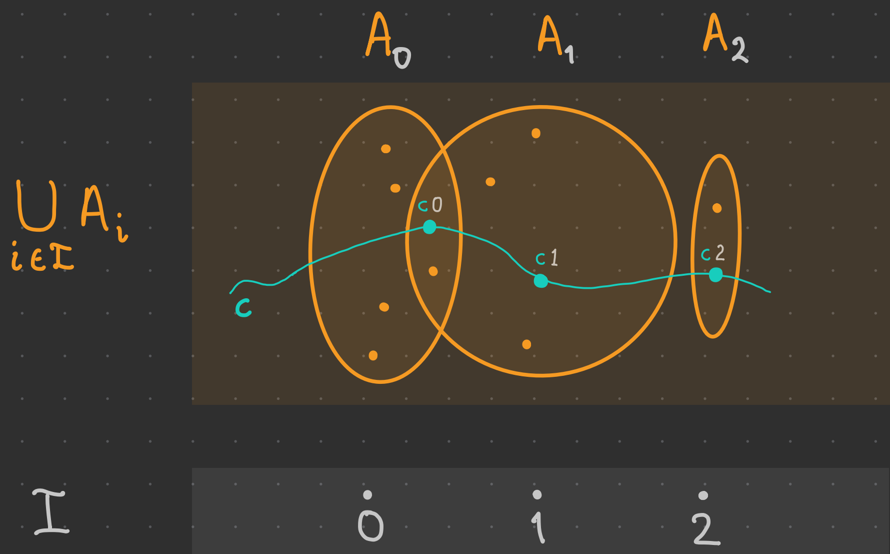
$$\bigcup_{i \in \{0,1\}} A_i \qquad (\forall i \in \{0,1\})[fi \in A_i]$$

função de escolha

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=}$$

$c \in \cancel{A_0 \times A_1 \times A_2}$

$\prod_{i \in \{0,1,2\}} A_i$



HW:

$e \circ \sum_{i \in I} A_i$?

Aritmética de tipos

$\alpha + \beta \;,\; \alpha \times \beta$

$\alpha^{\beta} \equiv \beta \to \alpha$

$0 \;,\; \mathbb{1}$

Know your sums!

$1 + \alpha \cong \text{Maybe } \alpha$

$\mathbb{B} \times \alpha \cong 2 \times \alpha \cong \alpha + \alpha$

Três leis/construções familiares

$\gamma^{\beta\alpha} \cong (\gamma^{\alpha})^{\beta}$ currificação

$\gamma^{\alpha}\gamma^{\beta} \cong \gamma^{\alpha+\beta}$ copairing

$\alpha^{\delta}\beta^{\delta} \cong (\alpha\beta)^{\delta}$ pairing

Aritmética com tipos infinitos

$$\infty + 1 \text{ “}\overset{?}{=}\text{” } \infty \ ?$$

$$\begin{tikzcd} \mathbb{N} + \mathbb{1} \arrow[rr, bend left, "F"] & \overset{?}{\cong} & \mathbb{N} \arrow[ll, bend left, "G"] \end{tikzcd}$$

$$F\ (l \cdot n) \overset{\text{def}}{\equiv} S\, n$$

$$F\ (r \cdot \_) \overset{\text{def}}{\equiv} 0$$

$$G\ 0 \overset{\text{def}}{\equiv} r \cdot \star$$

$$G\ (S\, n) \overset{\text{def}}{\equiv} l \cdot n$$

$$\infty + \infty \text{ “}\overset{?}{=}\text{” } \infty \ ?$$

$$\begin{tikzcd} \mathbb{N} + \mathbb{N} \arrow[rr, bend left, "F"] & \overset{?}{\cong} & \mathbb{N} \arrow[ll, bend left, "G"] \end{tikzcd}$$

$$F\ (l \cdot n) \overset{\text{def}}{\equiv} 2n$$

$$F\ (r \cdot n) \overset{\text{def}}{\equiv} 2n+1$$

$$\begin{aligned} G\ n \overset{\text{def}}{\equiv}\ & \text{if even } n \\ & \text{then } l \cdot (n \overset{\text{quot}}{/} 2) \\ & \text{else } r \cdot \big(\underbrace{(n-1) / 2}_{= n/2}\big) \end{aligned}$$

Cantor

lec15

2025-05-14

$\text{count} : \text{FinSet}\ \alpha \to \mathbb{N}$

$A, B : \text{FinSet}\ \alpha$

$A \overset{?}{=}_c B \rightsquigarrow \text{count}\ A \overset{?}{=}_{\mathbb{N}} \text{count}\ B$

$A =_c B \overset{\heartsuit}{\iff}$ A, B têm a mesma cardinalidade

A, B são equinúmeros

$A =_c B \overset{\text{def}}{\iff} (\exists f : A \to B)[\, f \text{ bij}\,] \overset{\text{def}}{\iff} (A \rightarrowtail\!\!\!\!\twoheadrightarrow B)$ hab.

$A \leq_c B \overset{\text{def}}{\iff} (\exists B' \subseteq B)[\, A =_c B' \,] \overset{?}{\iff} (\exists f : A \to B)[\, f \text{ inj}\,]$

$A <_c B \overset{\text{def}}{\iff} A \leq_c B \ \&\ A \neq_c B \overset{?}{\iff} (\exists B' \subsetneq B)[\, A =_c B' \,]$

aleph

$\aleph_0 \overset{\text{def}}{\equiv}$ a cardinalidade do $\mathbb{N}$

$\aleph_0 : \text{Card}$?

Investigação: $(=_c)$ & $(\leq_c)$

$(=_c), (\leq_c) : \mathrm{Set}\ ? \times \mathrm{Set}\ ? \to \mathrm{Prop}$

$(=_c)$ relação de equivalência?

$(\leq_c)$ relação de ordem?

$(=_c)$ compat com $\wp_f\_$? $\cup\_$? $\cap\_$?

[Qual pergunta deveria estar aqui?]

$(\to), (\times), (\uplus)$

powerset-finito: $\wp_f A \overset{\text{def}}{\equiv} \{ F \subseteq A \mid F \text{ fin} \}$

$\mathbb{Z}$

$\mathbb{N} =_c \mathbb{Z}$ pois $0, 1, -1, 2, -2, \ldots$

$f\ x \overset{\text{def}}{\equiv}\ ?$

$\mathbb{Q}$?

Dicas

lec16

2025-05-16

$(\le_c)$ relação de ordem?

[Essas perguntas deveriam estar aqui!]:

$$\frac{A \le_c A'}{F\ A \le_c F\ A'}\ ?$$

$$\frac{A \le_c A' \qquad B \le_c B'}{F\ A\ B \le_c F\ A'\ B'}\ ?$$

$$\frac{A \le_c A'}{F\ A\ B \underset{\ge_c}{\le_c} F\ A'\ B}\ ?$$

$$\frac{B \le_c B'}{F\ A\ B \underset{\ge_c}{\le_c} F\ A\ B'}\ ?$$

$(=_c)$-refl : $\dfrac{}{A =_c A}\ ?$

$(=_c)$-trans ; $\dfrac{A =_c B \qquad B =_c C}{A =_c C}\ ?$

$(=_c)$-sym : . . .

$(\le_c)$-refl . . .

$(\le_c)$-trans . .

~~$(\le_c)$-antisym.~~

$(\le_c)$-"equi antisym":

$$\frac{A \le_c B \qquad B \le_c A}{A =_c B}$$

Jogos [Smullyan]

$s \in \{1, 2, 3\}$

Dia 0: Será que $s = 1$?

Dia 1: Será que $s = 2$?

$s \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ (ou $\mathbb{Q}_{\geq 0}$)

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$
Dia 0:	0	0	?
Dia 1:	1	1	
Dia 2:	2	−1	
	⋮	2	
		−2	
		⋮	

C contável

ou (a)enumerável

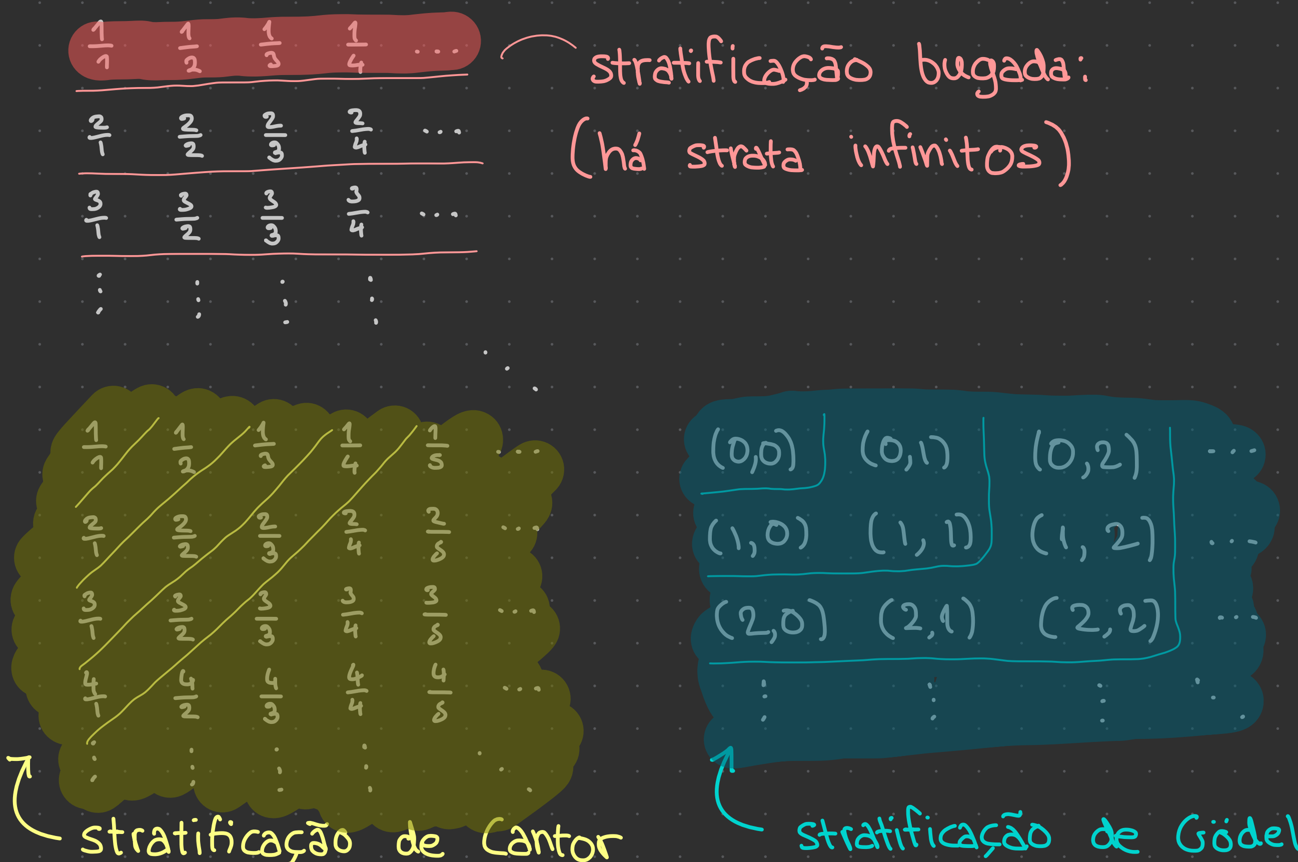
$$\Big\Updownarrow_{\text{def}}$$

$$C =_c \mathbb{N}$$
$$\leq_c$$

$$\Big\Updownarrow_{♡}$$

existe estratégia vencedora . .

Primeiro argumento diagonal de Cantor



Mais desafios

$$A^* \overset{\text{def}}{\equiv} \bigcup_{\ell=0}^{\infty} A^{\ell}$$

ou $A^{<\omega}$

(*) $\mathcal{P}_f \mathbb{N}$

$\emptyset$

$\{0\}$

$\{0,1\}$

$\{1\}$

$\vdots$

$\mathbb{N}^*$ — Kleene star

$[\,]$

$[0]$

$[1]$

$[0,0]$

$[0,1]$

$[1,0]$

$[1,1]$

$[2]$

$\vdots$

$\mathcal{P}_f \overbrace{\mathcal{P}_f \mathbb{N}}^{=_c \mathbb{N}}$

exemplo de S:

$\{\ \{1,2,8\},\ \{7000, \ldots, 7203\},\ \emptyset,\ \{43\}\ \}$

já resolvido (!) pois:

$$\dfrac{}{\mathcal{P}_f \mathbb{N} =_c \mathbb{N}}\ (*)$$

$$\dfrac{\mathcal{P}_f \mathbb{N} =_c \mathbb{N}}{\mathcal{P}_f \mathcal{P}_f \mathbb{N} =_c \mathcal{P}_f \mathbb{N}}\ \text{compat (HW anterior)}$$

Segundo argumento diagonal de Cantor

0,

7 2 1 0 0 2 1 ...
5 5 2 0 1 2 1 ...
7 0 0 0 0 0 0 ...
0 0 0 0 0 0 0 ...
9 9 9 9 9 9 9 ...
3 2 7 6 6 6 6 ...
3 3 3 3 3 3 3 ...
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋱

0, 8 6 1 1 0 7 4 ...

[0, 1] : Set Real

HW

(1) Qual é o problema?

(2) Corrija.

Diagonalização de Cantor

lec17

2025-05-19

$$
\begin{array}{lcccccccc}
0, & & & & & & & & \\
 & 8 & 6 & 1 & 1 & 0 & 7 & 4 & \dots \\
 & 7 & 2 & 1 & 0 & 0 & 2 & 1 & \dots \\
 & 5 & 5 & 2 & 0 & 1 & 2 & 1 & \dots \\
 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & \dots \\
 & 3 & 2 & 7 & 6 & 6 & 6 & 6 & \dots \\
 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0, & 5 & 5 & 5 & 5 & 5 & 5 & 5 & \dots
\end{array}
$$

Irracionais

$$\mathbb{R}_{\mathbb{Q}}$$

$\mathbb{Q}$ cont. $\subseteq$ $\mathbb{R}$ incont.

$$\underset{c}{\mathbb{Q}} \cup \underset{i}{\mathbb{I}} = \underset{i}{\mathbb{R}}$$

HW

$I \subseteq X$

I incontável

$\vdash X$ incontável

Usando a diagonalização

construimos (uma infinidade de) irracionais

(contável)

$\mathbb{A}$lgébricos

$f(x) = 2x^3 + 3x + 4$

e $g(x) = \frac{7}{2}x^2 + \frac{1}{2}x + 12$

a algébrico $\overset{\text{def}}{\iff}$ a raiz de algum $f \in \boxed{\mathbb{Z}}[x]$

$g \in \boxed{\mathbb{Q}}[x]$

tanto faz!

$R[x] \overset{\text{def}}{\equiv} \{$ polinômios de uma var x com coefs em $R \}$

t transcendental $\overset{\text{def}}{\iff}$ t não algébrico

$\mathbb{Q}[x]$ contável (dica: $f \mapsto (4, 3, 0, 2)$)

$\mathbb{A}$ contável (dica: $|\text{roots}_{\mathbb{R}}(f)| \leq \deg(f)$)

$\mathbb{A} = \bigcup_{f \in \mathbb{Q}[x]} \text{roots}(f)$

Θ. Cantor

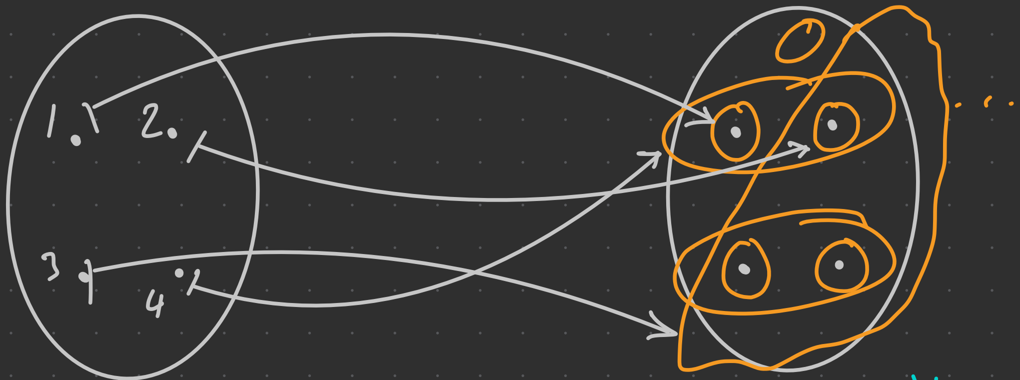
$$A <_c \wp A \iff \underbrace{A \leq_c \wp A}_{\text{wit: } \{\_\}} \;\&\; A \neq_c \wp A$$

singletonizer

Seja $\pi : A \to \wp A$.

Basta mostrar que π não é sobrejetiva.

Ou seja, procuro $Y \in \wp A$ t.q. $Y \notin \pi[A]$.



$\{ d \in A \mid d \text{ depressivo} \}$

$\overset{\text{def}}{=}$

Vou mostrar: $D \notin \pi[A]$.

$\pi\, a \equiv \{ y : A \mid a \text{ ama } y \}$

Seja t t.q. $\underbrace{t \overset{\pi}{\longmapsto} D}_{\pi t = D}$.

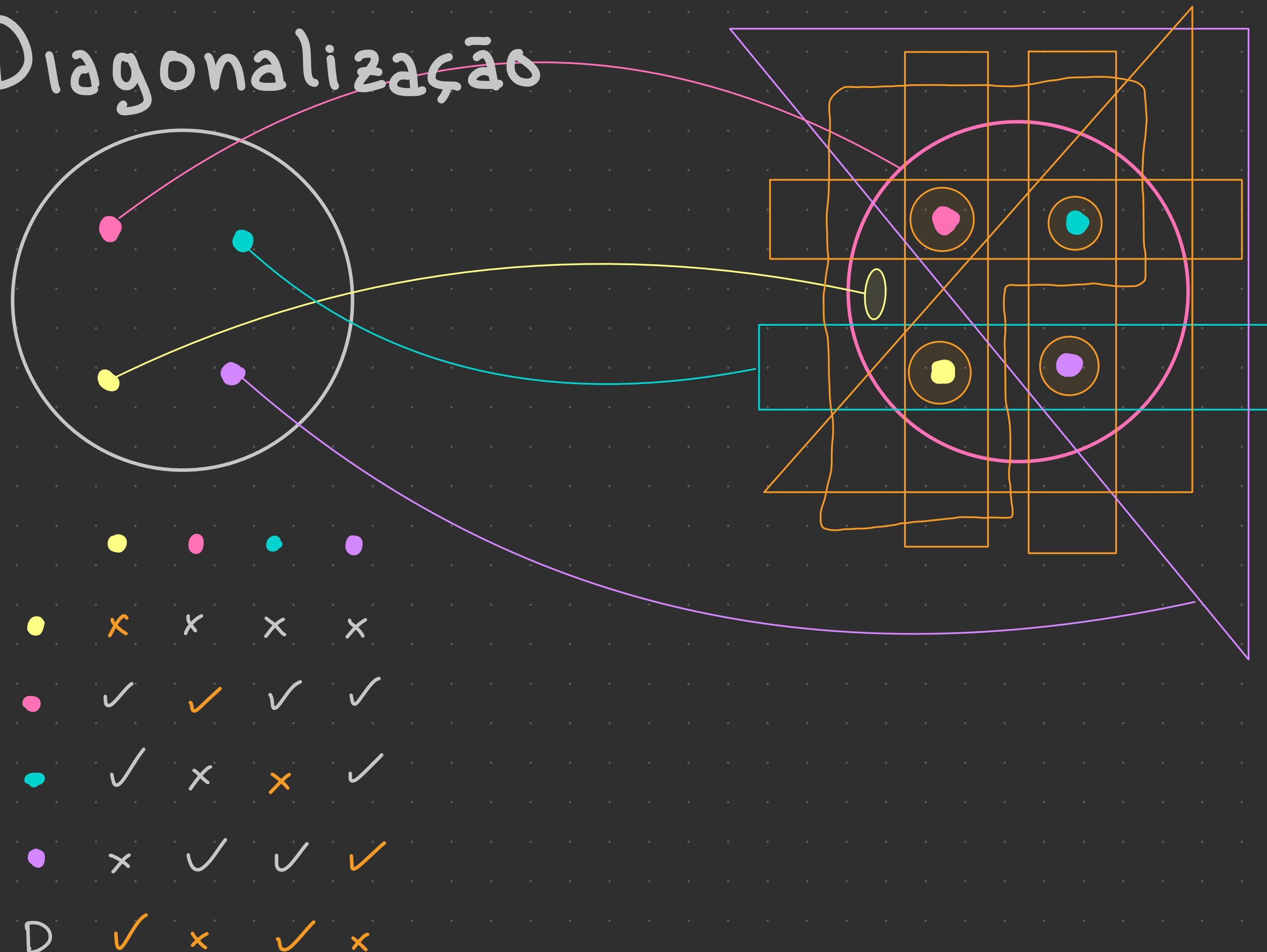
a narcisista $\overset{\text{def}}{\iff}$ $\pi\, a = \{a\}$

a misânthropo $\overset{\text{def}}{\iff}$ $\pi\, a = \emptyset$

a depressivo $\overset{\text{def}}{\iff}$ $a \notin \pi\, a$

$$\dfrac{\dfrac{\dfrac{t \text{ dep}}{t \notin \pi t}}{t \notin D}}{\dfrac{t \text{ não dep}}{\bot}} \qquad \dfrac{\dfrac{\dfrac{t \text{ não dep}}{t \in \pi t}}{t \in D}}{\dfrac{t \text{ dep}}{\bot}}$$

Diagonalização



Conseqüencia

$$\mathbb{N} <_c \wp\mathbb{N} <_c \wp\wp\mathbb{N} <_c \wp^3\mathbb{N} <_c \cdots$$

Procurando bijecções

$$(0,1) \xmapsto{\lambda x.2x} (0,2)$$

$$\begin{tikzcd} (2,3) \arrow[rr, mapsto, "?"] \arrow[dr, mapsto, bend right, "\text{shift }(-2)"'] & & (0,5) \\ & (0,1) \arrow[ur, mapsto, bend right, "\text{stretch }5"'] & \end{tikzcd}$$

HW

$$[0,1) \overset{?}{=}_c (0,1)$$

$$\left.\begin{array}{l} (\ ,\ ) \\ [\ ,\ ) \\ (\ ,\ ] \\ [\ ,\ ] \end{array}\right\} \text{todos } (=_c)$$

θ. Cantor-Schröder-Bernstein [CBS]

$$\frac{A \leq_c B \qquad B \leq_c A}{A =_c B}\ \text{CBS}$$

COROLÁRIOS: (HW anterior)

$$\frac{(0,1) \leq_c (0,1) \qquad (0,1) \leq_c [0,1)}{[0,1) =_c (0,1)}\ \text{CBS}$$

Dica anthropomórfica [Smullyan]

Planeta: $A \cup B$

(1) cada A ama exatamente uma B

(2) nenhuma B é amada por dois A

(3) cada B ama exatamente um A

(4) nenhum A é amado por duas B

Objetivo: achar uma maneira de casar todos

- monogâmicos
- heterosexuais
- em cada casal existe amor (não necessariamente recíproco)

Casamentos (dica)

lec18

2025-05-26

Considere pessoa x. Pergunte: Quem te ama?

$$(x_i)_i : \text{Seq Person} \quad \begin{cases} x_0 \overset{\text{def}}{\equiv} x \\ x_1 \overset{\text{def}}{\equiv} \text{aquela pessoa que ama } x_0 \\ x_{n+1} \overset{\text{def}}{\equiv} \text{aquela pessoa que ama } x_n \end{cases}$$

$$C_\infty \overset{\text{def}}{\equiv} \{ x \mid \text{path}(x) \text{ infinito} \}$$

$$C_A \overset{\text{def}}{\equiv} \{ x \mid \text{path}(x) \text{ termina em } A \}$$

$$C_B \overset{\text{def}}{\equiv} \{ x \mid \text{path}(x) \text{ termina em } B \}$$

Hypergame

- Jogo terminante

- Hypergame : Player 1 : escolher um jogo terminante

P1 : Jogo da velha

P2 : 

P1 : 

⋮

P1 : Hypergame

P2 : Hypergame

P1 : Hypergame

⋮

partida infinita de Hypergame

- Hypergame é terminante

mas então

HW Outra demonstração de $\ominus$ Cantor: $A <_c \wp A$

Codificações

$$\mathbb{Q}_{>0} =_c \mathbb{N} \qquad \frac{3}{4} \longmapsto 1000100001$$

$$\mathbb{Q} =_c \mathbb{N} \qquad \frac{-2}{3} \longmapsto 120010001$$

$$\mathbb{N}^* =_c \mathbb{N} \qquad (4,0,2) \longmapsto 1000011001$$

Continuum Hypothesis (CH)

$\mathbb{N} \quad <_c \quad p\mathbb{N}$

$<_c \;?\; <_c$

Generalized C.H.

$A \quad <_c \quad pA$

$<_c \;?\; <_c$

Cardinal comparability

A, B : Set $\overset{?}{\vdash}$ $A \leq_c B$ ou $B \leq_c A$

Russell

Seja $R \stackrel{\text{def}}{\equiv} \{ x \mid x \notin x \}$.

$$\dfrac{\dfrac{R \in R}{R \notin R}}{\perp} \qquad \dfrac{\dfrac{R \notin R}{R \in R}}{\perp} \quad (?)$$