Mundos matemáticos

lec01
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Exemplos:

tipos/conjuntos — funções, injeções, sobrejeções } 3 mundos

inteiros — (|), (≤) } 2 mundos

$$\begin{tikzcd} 2 \arrow[r, bend left, "|"] & 8 \arrow[l, bend left, "\not"] \end{tikzcd}$$

grupos — homomorfismos

aneis — hom..

monóides — hom...

posets — monótonas (isótonas)

reticulados — ?

lattices — hom...

Set α (⊆)

$$\begin{tikzcd} \{2\} \arrow[r, bend left, "\subseteq"] \arrow[rr, bend right, "\subseteq"] & \{2,5\} \arrow[r, bend left, "\subseteq"] & \text{Primes} \arrow[r, bend left, "\not"] & \text{Even} \end{tikzcd}$$

Cats (Categorias)

$\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, ...

$\mathbf{C}$

$\mathcal{C}$

$\mathbb{C} \equiv$ tambem $\mathbb{C}_0$

- $\mathrm{Obj}(\mathbb{C})$: A, B, C, ...
- $\mathrm{Arr}(\mathbb{C})$: f, g, h, ... também $\mathbb{C}_1$

- também: $\mathrm{Arr}(A,B)$, $\mathrm{Hom}(A,B)$

  $\mathbb{C}(A,B) \equiv \{$ as setas de A para $B \}$

- também: $(;)_{A \to B \to C} \equiv \mathrm{swap}(\circ)$

  $(\circ)_{A \to B \to C} : \mathbb{C}(B,C) \times \mathbb{C}(A,B) \to \mathbb{C}(A,C)$

- $1_A : \mathbb{C}(A,A)$ também: id_A

$\underset{\text{src}}{\text{dom}} : \mathrm{Arr}(\mathbb{C}) \to \mathrm{Obj}(\mathbb{C})$

$\underset{\text{tgt}}{\text{cod}} : \mathrm{Arr}(\mathbb{C}) \to \mathrm{Obj}(\mathbb{C})$

Escrevemos:

$A \xrightarrow{f} B$ ou $f : A \longrightarrow B$

para: $A, B \in \mathrm{Obj}(\mathbb{C})$

& $f \in \mathrm{Arr}(\mathbb{C})$

& $\mathrm{src}(f) = A$

& $\mathrm{tgt}(f) = B$

Leis:

(∘) - assoc

$$\underbrace{(\forall A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D)}_{(\forall A \cdot Obj(\mathbb{C}))\cdots(\forall f : \mathbb{C}(A,B))}[(h\circ g)\circ f = h\circ(g\circ f)]$$

$$\underbrace{(h\,g)\,f = h\,(g\,f)}$$

tendo apenas uma op binária aproveitamos justaposição gratuitamente

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& C \arrow[r, "h"'] & D
\end{tikzcd}$$

comuta

$(1_?)$ - id - (∘)

$$(\forall A \xrightarrow{f} B)[\,1_B \circ f = f = f \circ 1_A\,]$$

$(\forall D \xrightarrow{f} A)\ [\,1_A \circ f = f\,]$

$(\forall A \xrightarrow{f} C)\ [\,f \circ 1_A = f\,]$

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[dr, bend right, "f"'] & A \arrow[d, "f"] \arrow[dr, bend left, "f"] & \\
& B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

comuta

Construções

$\mathbb{1}$

$$\begin{tikzcd} \star \arrow[loop left, "1_\star"] \end{tikzcd}$$

$\mathbb{2}$

$$\begin{tikzcd} \star \arrow[r, bend left, "f"] \arrow[loop below, "1_\star"] & \bullet \arrow[loop below, "1_\bullet"] \end{tikzcd}$$

$\mathbb{3}$? $\mathbb{4}$? $\mathbb{n}$?

$1_\star \circ 1_\star \;:\; \mathbb{C}(\star, \star)$

$f \circ 1_\star \;:\; \mathbb{C}(\star, \bullet)$

$\mathbb{0}$

$\overline{\mathbb{2}}$

$$\begin{tikzcd} \star \arrow[loop below, "1_\star"] & \bullet \arrow[loop below, "1_\bullet"] \end{tikzcd}$$

$$\begin{tikzcd} \star \arrow[r, bend left, "f"] & \bullet \arrow[l, bend left, "g"] \end{tikzcd}$$

mas $f \circ g = ?$

$g \circ f = ?$

oposta ou dual

$\mathbb{C} : \mathbf{Cat} \rightsquigarrow \mathbb{C}^{op} : \mathbf{Cat}$

$.\mathrm{Obj} :\equiv \mathrm{Obj}(\mathbb{C})$

$.\mathrm{Arr} :\equiv A^{op} \xrightarrow{f^{op}} B^{op} \iff A \xleftarrow{f} B$

$1^{op}_{A^{op}} \equiv 1_A \qquad f^{op} \circ^{op} g^{op}$

$\mathbb{C}$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[dr, bend right, "g \circ f"'] & B \arrow[d, bend left, "g"] \\ & C \end{tikzcd}$$

$\mathbb{C}^{op}$

$$\begin{tikzcd} A & B \arrow[l, bend right, "f^{op}"'] \\ & C \arrow[u, bend right, "g^{op}"'] \arrow[ul, bend left, "f^{op} \circ^{op} g^{op} \equiv (g \circ f)^{op}"] \end{tikzcd}$$

a partir dum proset

$\mathcal{P} : \mathbf{P(r)oset} \rightsquigarrow \mathbb{C}[\mathcal{P}] : \mathbf{Cat}$

$.\mathrm{Obj} :\equiv \mathcal{P}.\mathrm{carrier}$

$.\mathrm{Arr} :\equiv A \to B \iff A \leq_p B$

escrevemos também

$A \xrightarrow{\leq} B$

Caso $A \leq_p B$: $\mathbb{C}(A,B)$ singleton

Caso $A \not\leq_p B$: $\mathbb{C}(A,B)$ vazio

Mais construções

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a partir dum monoid

$\mathcal{M} : \text{Monoid} \rightsquigarrow \mathbb{C}[\mathcal{M}] : \text{Cat}$

$\mathcal{M} \equiv (M; \cdot_{\mathcal{M}}, u_{\mathcal{M}})$

$.\mathrm{Obj} :\equiv \{\star\}$

$.\mathrm{Arr} :\equiv \mathcal{M}.\mathrm{carrier}$

$1_\star :\equiv u$

$f \circ g :\equiv f \cdot_{\mathcal{M}} g$

Definições categoriais: Propriedades universais

T terminal $\overset{def}{\iff}$ $\forall X \xrightarrow{\exists!} T$ $\iff$ $(\forall x)(\exists! f)[f : X \to T]$

I coterminal
inicial $\overset{def}{\iff}$ $\forall X \xleftarrow{\exists!} I$

$\overset{\ell}{\swarrow} P \overset{r}{\searrow}$ produto dos A, B $\overset{def}{\iff}$

$$\begin{tikzcd}
A & P \arrow[l, "\ell"'] \arrow[r, "r"] & B \\
& \forall C \arrow[ul, bend left, "\forall f"] \arrow[ur, bend right, "\forall g"'] \arrow[u, dotted, "\exists!"] &
\end{tikzcd}$$

$A \times B$

$\overset{\ell}{\nearrow} S \overset{r}{\nwarrow}$ coproduto
soma dos A, B $\overset{def}{\iff}$

$$\begin{tikzcd}
A \arrow[r, "\ell"] \arrow[dr, bend right, "\forall f"'] & S \arrow[d, dotted, "\exists!"] & B \arrow[l, "r"'] \arrow[dl, bend left, "\forall g"] \\
& \forall C &
\end{tikzcd}$$

$A + B$

Cuidado: não temos unicidade (literalmente). Por exemplo:

$$\begin{tikzcd} & \forall C \arrow[dl, bend left, "\forall f"] \arrow[d, dotted, "\exists !"] \arrow[dr, bend right, "\forall g"'] & \\ A & A \times B \arrow[l, "\mathrm{outl}"'] \arrow[r, "\mathrm{outr}"] & B \end{tikzcd}$$

ambos merecem o nome produto dos A, B

$$\begin{tikzcd} & \forall C \arrow[dl, bend left, "\forall f"] \arrow[d, dotted, "\exists !"] \arrow[dr, bend right, "\forall g"'] & \\ A & B \times A \arrow[l, "\mathrm{outr}"'] \arrow[r, "\mathrm{outl}"] & B \end{tikzcd}$$

MAS: temos a "unicidade" que importa:

Θ. T, T' terminais $\vdash T \cong T'$

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[dr, "1"'] & T' \arrow[d, "f\ !"] \arrow[dr, "1"] & \\ & T \arrow[r, "!", "f'"'] & T' \end{tikzcd}$$

$$\frac{T \text{ terminal}}{T' \xrightarrow[f]{!} T}\,[X := T'] \qquad \frac{T' \text{ terminal}}{T \xrightarrow[f']{!} T'}\,[X := T]$$

$f \circ f' \overset{?}{=} 1_T \qquad f' \circ f \overset{?}{=} 1_{T'}$ HW

$$\frac{\dfrac{T \text{ terminal}}{T \xrightarrow{!} T}\,[X := T] \qquad T \xrightarrow{f \circ f'} T \qquad T \xrightarrow{1} T}{f \circ f' = 1}$$

Propriedades de setas

$f : A \to B$ (ou $\mathbb{C}(A,B)$, ou $\mathrm{Arr}(A,B)$)

f mono	$\overset{\text{def}}{\iff}$	$(\circ)$-cancelável-L
f epi		$(\circ)$-cancelável-R
f split mono		$(\circ)$-invertível-L
f split epi		$(\circ)$-invertível-R
f endo		$A = B$ $(\mathrm{src}(f) = \mathrm{tgt}(f))$
f iso		$(\circ)$-invertível
f auto		iso + endo

$A \cong B \overset{\text{def}}{\iff} (\exists f : A \to B)[\, f \text{ iso} \,]$ $\qquad \xrightarrow{\cong}$, $f : A \cong B$, ...

Traduzindo definições categoriais

… na $\mathbb{C}[(\mathbb{Z};(|))] : \mathsf{Cat}$

não oferece algo neste mundo (por quê?)

I inicial $\iff I$ obj t.q. $(\forall X \text{ obj})(\exists! f \text{ arr})[f : I \to X]$

$\iff I \in \mathbb{Z}$ t.q. $(\forall n \in \mathbb{Z})[I \mid n]$

T terminal $\iff$ . .

conclusão:
iniciais: $1, -1$
terminais: 0

${}_{\swarrow}^{\ell} P {}^{r}_{\searrow} = A \times B \iff P$ obj, $\ell : P \to A$, $r : P \to B$

$(\forall A \xleftarrow{f} C \xrightarrow{g} B)(\exists! h : C \to P)[\dots]$

$\iff P : \mathbb{Z}$, $\ell : P \mid A$, $r : P \mid B$

$(\forall C$ div com $A, B)[C \mid P]$

conclusão:
produtos: mdc

$\iff P$ é um m.d.c. dos A, B

Terminais e iniciais na...

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Set

$A \xrightarrow{!} \{*\}$

terminal: os singletons $\{*\}$, $\{1\}$, $\{\text{Brasil}\}$, ...

inicial: $\varnothing \xrightarrow{!} A$

Group

terminal: $\{e\} \xleftarrow{!} G$

inicial: $\{e\} \xrightarrow{!} G$

é um objeto zero
null

$$\begin{tikzcd} & ☺ \arrow[dr, bend left, "!"] & \\ A \arrow[ur, bend left, "!"] \arrow[rr, "!0!"'] & & B \end{tikzcd}$$

... portanto a presença de ☺ garanta setas de cada obj para cada obj

(todos os os hom-sets são habitados)

• • •

Cat

$\mathrm{Obj}(\underline{\mathrm{Cat}})$:= as categorias (pequenas)

$\mathrm{Arr}(\underline{\mathrm{Cat}})$:= os functors

F functor (covariant)

$F_0 : \mathbb{C}_0 \to \mathbb{D}_0$

$F_1 : \mathbb{C}_1 \to \mathbb{D}_1$

$$\frac{A \xrightarrow{f} B}{FA \xrightarrow{Ff} FB}$$

contravariant

$$\frac{A \xrightarrow{f} B}{FA \xleftarrow{Ff} FB}$$

ou seja: $\begin{cases} src \circ F_1 = F_0 \circ src \\ tgt \circ F_1 = F_0 \circ tgt \end{cases}$

?

leis: $\boxed{\begin{array}{l} F_1 1_A =_{\mathbb{D}_1} 1_{F_0 A} \\ F_1(f \circ g) =_{\mathbb{D}_1} F_1 f \circ F_1 g \end{array}}$

?

Exemplos

powerset (imagem; pré-imagem) $\quad \underline{\mathbf{Set}} \to \mathbf{Set}$

$$\frac{f : A \longrightarrow B}{\vec{\wp} f : \vec{\wp} A \to \vec{\wp} B} \qquad \frac{f : A \longrightarrow B}{\overleftarrow{\wp} f : \overleftarrow{\wp} A \leftarrow \overleftarrow{\wp} B}$$

HW: verifique!

Forgetful

exemplos: $\underline{\mathbf{Group}} \xrightarrow{U} \underline{\mathbf{Monoid}} \xrightarrow{U} \underline{\mathbf{Semigroup}} \xrightarrow{U} \underline{\mathbf{Magma}} \xrightarrow{U} \underline{\mathbf{Set}}$

Hask (mentira)

- Obj := tipos
- Arr := programas (funções) $\alpha \to \beta$

List/map

$$\text{List} : \text{Hask} \longrightarrow \text{Hask}$$

portanto um endofunctor

wow!

$$
\begin{tikzcd}
\text{Int} \arrow[r, bend left, "\text{List}"] \arrow[d, "ev"'] & \text{List Int} \arrow[d, "\text{List } ev"] \\
\text{Bool} & \text{List Bool}
\end{tikzcd}
$$

(Haskelleiros: fmap ev)

leis:

$$\text{map } id_{\text{Int}} = id_{\text{List Int}}$$

$$\text{map } (f \circ g) = \text{map } f \circ \text{map } g$$

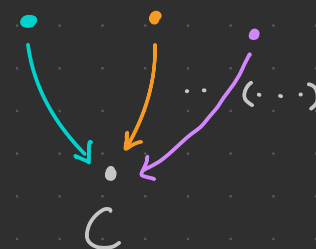
Mais construções

dualize para obter as **under** (aka coslice)

Over categorias (aka slice)

$$\mathbb{C} : \mathrm{Cat}\,, \quad C : \mathbb{C}_0 \;\longmapsto\; \mathbb{C}/C \;\textbf{ ou }\; \begin{matrix}\mathbb{C}\\ \downarrow \\ C\end{matrix} \;:\; \mathrm{Cat}$$

. Obj $:= \bigcup_{X \in \mathbb{C}_0} \mathrm{Arr}(X, C)$



. Arr := os triângulos

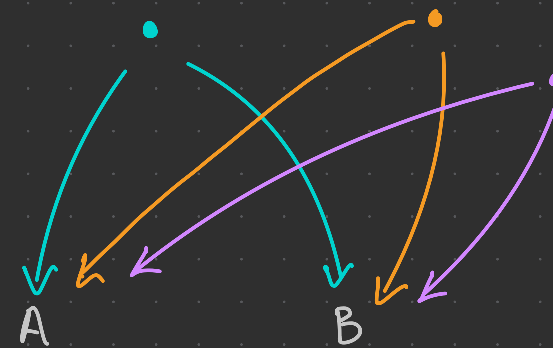
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "="] & Y \arrow[dl] \\ & C & \end{tikzcd}$$

verifique:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[drr] & & Y \arrow[rr, "g"] \arrow[d] & & Z \arrow[dll] \\ & & C & & \end{tikzcd}$$

$\mathbb{C}$: Cat, $A, B : \mathbb{C}_0$ $\longmapsto$ uma cat cujos objetos são os coloridos:

$A \times B$ na $\mathbb{C}$ é apenas™ um terminal dessa



(Ganhamos unicidade de produtos de graça!)

(o que falta fazer?)

E o $A + B$?

Transformações naturais

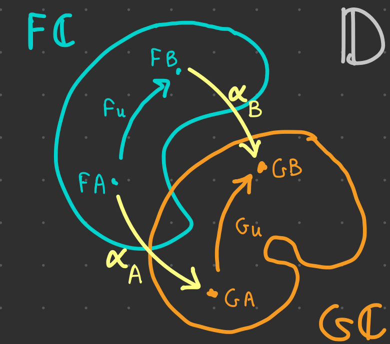
$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left=3, "F"] \arrow[r, shift right=3, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{D} \end{tikzcd}$$

$F, G : \mathbb{C} \longrightarrow \mathbb{D}$

$\alpha : F \Longrightarrow G$

o que deveria ser?

Uma família de setas:



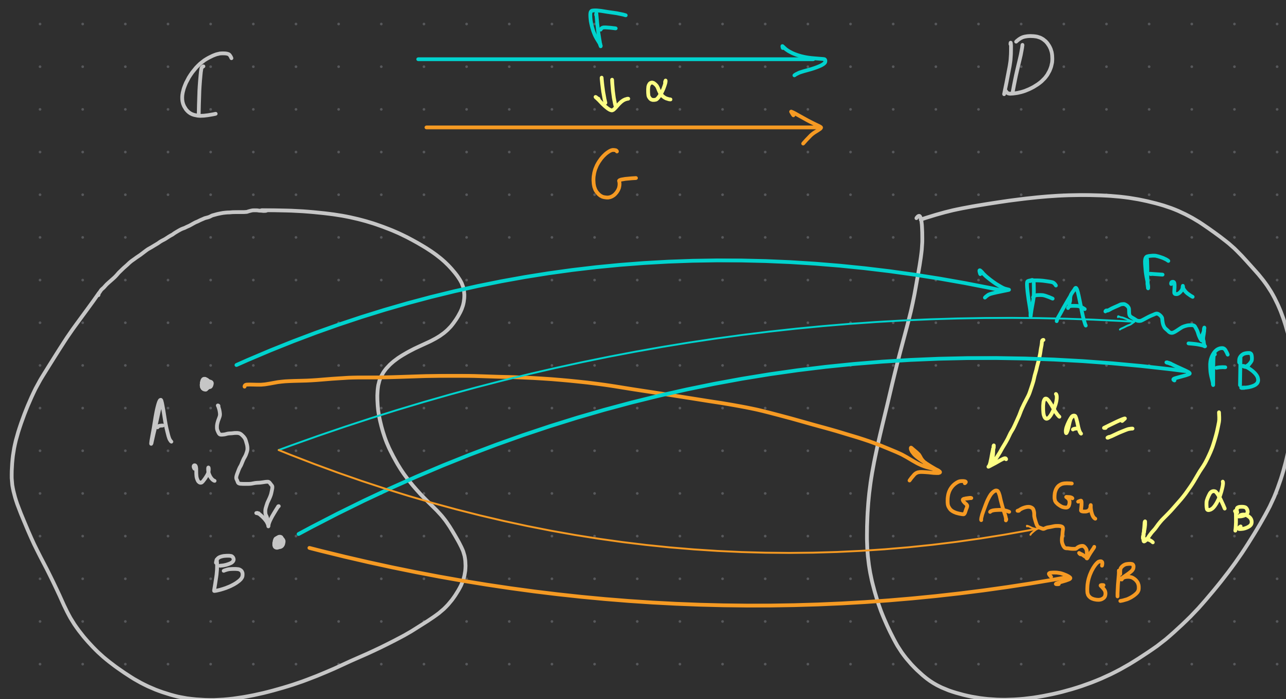
... indexada por ...?

Temos uma "correspondente" ao A, uma ao B, etc., ou seja, uma para cada objeto de $\mathbb{C}$, ou seja:

$$\alpha \equiv \left(\alpha_A : FA \to GA\right)_{A \in \mathbb{C}_0} \quad : \text{ família indexada pelos obj da } \mathbb{C}$$

lei:

$$\left(\forall \begin{tikzcd} A \arrow[d, "u"'] \\ B \end{tikzcd}\right) \left[\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Fu"'] \arrow[dr, phantom, "="] & GA \arrow[d, "Gu"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}\right]$$



$\mathbb{C}$
F
$\Downarrow \alpha$
G
$\mathbb{D}$
A
u
B
FA
Fu
FB
α_A
=
α_B
GA
Gu
GB