

# Os racionais

FTA :  $\text{Int} \rightsquigarrow \text{Rat}$

$$x : \text{Int} \xrightarrow{\text{FTA}} p_1^{a_1} \cdots p_{n_x}^{a_{n_x}} : \text{List Nat}$$

$$r : \text{Rat} \xrightarrow{\text{FTA}} p_1^{c_1} \cdots p_{n_r}^{c_{n_r}} : \text{List Int}$$

$$r = \frac{x}{y} = \frac{p_1^{a_1} \cdots p_{n_x}^{a_{n_x}}}{p_1^{b_1} \cdots p_{n_y}^{b_{n_y}}} = p_1^{a_1 - b_1} \cdots p_n^{a_n - b_n}$$

$$\text{e.g. } \frac{24}{42} = \frac{2^3 \cdot 3^1 \cdot 5^0 \cdot 7^0}{2^1 \cdot 3^1 \cdot 5^0 \cdot 7^1} = 2^{3-1} \cdot 3^{1-1} \cdot 5^{0-0} \cdot 7^{0-1} = 2^2 \cdot 3^0 \cdot 5^0 \cdot 7^{-1}$$

# Potências

$$a : \text{Int} \quad n : \text{Nat}$$

---

$$a^n : \text{Int}$$

$$a : \text{Rat} \quad x : \text{Int}$$

---

$$a^x : \text{Rat}$$

↪ e 0 0<sup>-1</sup> ?

# Irracionais

Ache  $x : ?$  t.q.  $x^2 = ? 2 : ?$

Seja  $r : \text{Rat}$  t.q.  $r^2 =_{\text{Rat}} 2 : \text{Rat}$   $r^2 = 2 \leadsto \left(\frac{u}{v}\right)^2 = 2 \leadsto u^2 = 2v^2$

Sejam  $u, v : \text{Int}$  t.q.  $u^2 =_{\text{Int}} 2v^2 : \text{Int}$

Logo  $u^2$  par.

Logo  $u$  par. [Lemma]

Logo seja  $k$  t.q.  $u = 2k$ .

Logo  $(2k)^2 = 2v^2$ .

Logo  $4k^2 = 2v^2$ .

Logo  $2k^2 = v^2$ .

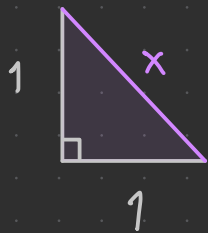
Logo  $v$  par. [mesmo Lemma]

Conclusão:

$u^2 = 2v^2 \Rightarrow u, v$  pares

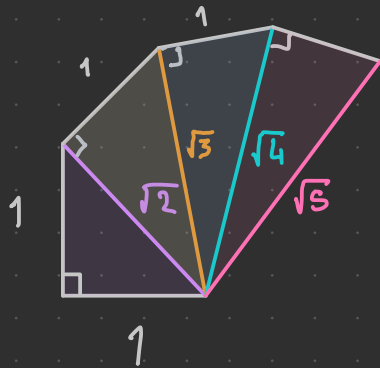
...E agora?  $\perp$ ?

# A outra metade do argumento



$$x^2 = 1^2 + 1^2$$

||  
2





# $\mathbb{O}_s$ reais (v1/?)

$(\mathbb{R}; 0, 1, +, -, \cdot)$

$0, 1 \in \text{Real}$

$+, \cdot : \text{Real} \times \text{Real} \rightarrow \text{Real}$

$- : \text{Real} \rightarrow \text{Real}$

Axiomas:

$\mathbb{R}A$ -ass

- id

- com

- inv

$\mathbb{R}M$ -ass

- id

- com

- inv\*

$(\forall a) [a \neq 0 \Rightarrow (\exists \dots$

$(\forall a \neq 0)(\exists a') [a' \text{ é } (\cdot)\text{-inv de } a]$

$$\mathbb{R}\text{-dist} : \begin{cases} d(a+b) = da + db \\ (a+b)d = ad + bd \end{cases}$$

# Teoria dos reais (v1/?)

uni

(+)-id

(+)-inv

(·)-id

(·)-inv

$(\forall l)(\forall l') [ l, l' \text{ reais} \Rightarrow l = l' ]$

$(\forall l, l' \text{ reais}) [ l = l' ]$

res

$$a + 0 = b$$

(\*)

$$a \cdot 1 = b$$

can

$$a + c = b + c \Rightarrow a = b$$

(\*)

$$a \cdot c = b \cdot c \Rightarrow a = b$$

ann

$$a \cdot 0 = 0$$

NZD

$$ab = 0 \Rightarrow a = 0 \text{ ou } b = 0$$

# Sugar

$$a - b \stackrel{\text{sug}}{\equiv} a + (-b)$$

$$a^{-1} \stackrel{\text{sug}}{\equiv} \text{o \u00fanico } (\cdot)\text{-inv de } a$$

$$a/b \stackrel{\text{sug}}{\equiv} a \cdot b^{-1}$$

$$a^{-n} \stackrel{\text{def}}{\equiv} \left\{ \begin{array}{l} (a^n)^{-1} \\ \parallel \Theta \\ (a^{-1})^n \end{array} \right.$$

$$a^0 \equiv 1$$

$$a^{n+1} \equiv \left\{ \begin{array}{l} a^n \cdot a \\ a \cdot a^n \end{array} \right\} \Theta \text{ Tanto faz!}$$



# Tipos dependentes (teaser)

$$\_ \_ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

função parcial (pode não retornar nada)

$$\_^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

$$\_^{-1} : ((a : \mathbb{R}) \times a \neq 0) \rightarrow \mathbb{R}$$

$$\_^{-1} : \mathbb{R} \times ? \rightarrow \mathbb{R}$$

Prop

$a \neq 0$

$$\_ / \_ : (\mathbb{R} \times (b : \mathbb{R}) \times b \neq 0) \rightarrow \mathbb{R}$$

quem é a?

"tuplas dependentes"

$$\left. \begin{array}{l} (3, (2, 0, 1)) \\ (2, (7, 7)) \\ (5, (3, 1, 1, 8, 3)) \end{array} \right\} : (n : \mathbb{N}) \times \mathbb{R}^n$$

"funções dependentes"

$$\begin{aligned} f & : (n : \mathbb{N}) \rightarrow \mathbb{R}^n \\ f \ 2 & = (3, 1) \\ f \ 4 & = (\sqrt{2}, 1, \pi, 3) \end{aligned}$$

# Uma grande mentira



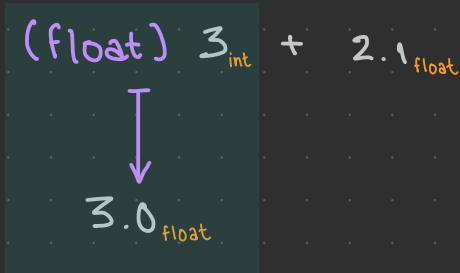
type errors!

$(\subseteq_{\alpha}) : \text{Set } \alpha \times \text{Set } \alpha \rightarrow \text{Prop}$

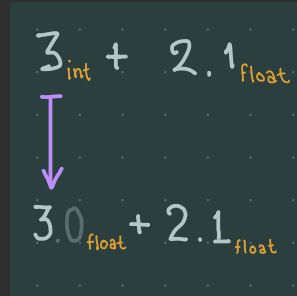
$3 : \text{Nat} \in \mathbb{Z}$

$(=_{\alpha}) : \alpha \times \alpha \rightarrow \text{Prop}$

type casting



type coercion



# Conjuntos "sistemas" de reais

$\mathbb{R}_{\mathbb{N}}$  <sup>def</sup>  $\{0, 1, 2, \dots\}$  : Set Real

$\mathbb{R}_{\mathbb{Z}}$   $\mathbb{R}_{\mathbb{N}} \cup -\mathbb{R}_{\mathbb{N}}$

$\mathbb{R}_{\mathbb{Q}}$   $\{x/y \mid x \in \mathbb{R}_{\mathbb{Z}}, y \in \mathbb{R}_{\mathbb{Z}} \setminus \{0\}\}$

$\mathbb{R}_A$   $\{x \mid x \text{ é raiz de algum polinômio no } \mathbb{Z}[x]\}$

os reais algébricos

$-A \stackrel{\text{def}}{=} \{-a \mid a \in A\}$

$A \setminus B \stackrel{\text{def}}{=} \{a \in A \mid a \notin B\}$



polinômios de uma variável  
com coeficientes inteiros

# Os reais ( $\sqrt{2}/?$ )

lec 03

2025-01-15

$(\mathbb{R}; +, 0, -, \cdot, 1, >)$

⋮

$(>) : \text{Real} \times \text{Real} \rightarrow \text{Prop}$

RO-Trans :  $a > b \ \& \ b > c \Rightarrow a > c$

- Tri e.u.d.:  $a > b ; a = b : b > a$

- A  $a > b \Rightarrow a + c > b + c$

- M\*  $a > b \Rightarrow c > 0 \Rightarrow ac > bc$

Sugar :  $(\leq), (\geq), (<), \text{Pos}, \text{Neg}, \dots$

Teoremas!

# Minima, maxima

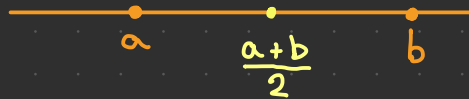
min, max : Real  $\times$  Real  $\rightarrow$  Real

: def


$$\begin{aligned}\min(a,b) &= \frac{a+b - |b-a|}{2} \\ &= \frac{a+b}{2} - \frac{1}{2}d(a,b)\end{aligned}$$

$$\begin{aligned}\max(a,b) &= \frac{a+b + |b-a|}{2} \\ &= \frac{a+b}{2} + \frac{1}{2}d(a,b)\end{aligned}$$

Teoremas!



# Valor absoluto

$|\_ | : \text{Real} \rightarrow \text{Real}$  

$$|x| \stackrel{\text{def}}{=} \begin{cases} x, & \text{caso } x \geq 0; \\ -x, & \text{caso } x < 0. \end{cases}$$

Teoremas!

# Distância

$$d : \text{Real} \times \text{Real} \rightarrow \text{Real}$$

melhor  
↓

$$d : \alpha \times \alpha \rightarrow \text{Real}$$

pré-distância

$$d(x, y) \geq 0$$

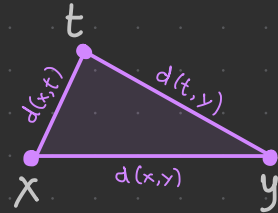
$$d(x, y) \leq d(x, t) + d(t, y)$$

$$d(x, y) = d(y, x)$$

$$d(x, x) = 0$$

$$\left( \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y \right)$$

$$d(x, y) = 0 \Rightarrow x = y$$



(triangular)

# Distâncias nos reais

Distância euclideana:

$$d(x, y) \stackrel{\text{def}}{=} |x, y|$$

$$\Theta. \quad d(x, y) \leq d(x, t) + d(t, y)$$

⋮

$$\text{ALVO:} \quad |x - y| \leq |x - t| + |t - y|$$

$$|x - y| = |(x + 0) - y| = |(x + (t + (-t))) - y|$$

$$= |x + t - t - y|$$

$$= |(x - t) + (t - y)|$$

$$\leq |x - t| + |t - y| \quad [\Theta.: |x + y| \leq |x| + |y|]$$

Verifique o resto!



Distância discreta:

$$d(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{caso } x = y; \\ 1, & \text{caso } x \neq y. \end{cases}$$

HW: ...é uma distância mesmo.

# Conjuntos

lec04  
2025-01-20

$\{0, 3'\}$  : Set Real

$\{\{0, \frac{1}{2}\}, \{42\}\}$  : Set (Set Real)

$x \in A \stackrel{\text{def}}{\iff} ?$

$\{x \mid \underbrace{\dots x \dots}_{\varphi(x)}\}$  : Set Real

$\varphi : \text{Real} \rightarrow \text{Prop}$

$\{w_p \mid p \in P\}$  : Set Real

$P : \text{Set Person}$

$w : \text{Person} \rightarrow \text{Real}$

# Intervalos

$$(a, b) \equiv \{ x \mid a < x < b \}$$

$$(a, +\infty) \equiv \{ x \mid a < x \}$$

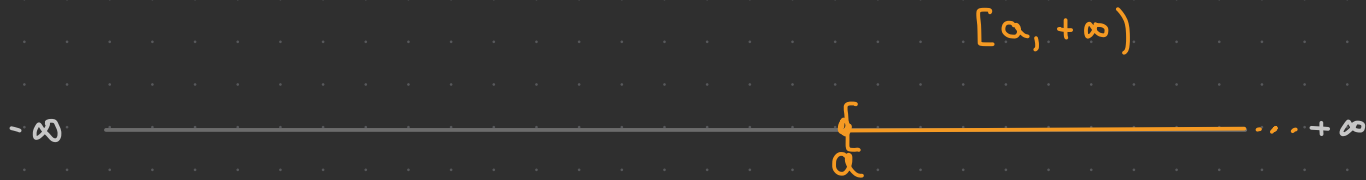
$$[a, b] \equiv \{ x \mid a \leq x \leq b \}$$

$$[a, +\infty), (-\infty, a), (-\infty, a]$$

$$[a, b) \equiv \{ x \mid a \leq x < b \}$$

$$(-\infty, +\infty) \equiv \mathbb{R}$$

$$[a, b] \equiv \{ x \mid a \leq x \leq b \}$$



# min, max

min, max : Set Real  $\xrightarrow{\text{yellow arrow}}$  Real

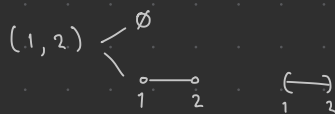
$\_ = \min \_$   
 $\_ = \max \_$  : Real  $\times$  Set Real  $\rightarrow$  Prop

$x = \min A \xLeftrightarrow{\text{def}} x \leq A \ \& \ x \in A$

$$\Updownarrow$$
$$(\forall a \in A)[x \leq a]$$

$\min A < \min B \Leftrightarrow (\exists a)(\exists b)[a = \min A \ \& \ b = \min B \ \& \ a < b]$

$\min A + \min B < \max C \Leftrightarrow (\exists a, b, c)[a = \min A \ \& \ \dots]$



# Floors & ceilings

$\lceil \ ]$   
↑  
floor  
piso

$\lfloor \ ]$   
↑  
ceiling  
teto

$\lfloor \ ]_{\text{Int}} : \text{Real} \rightarrow \text{Int}$

$\lfloor \ ]_{\text{Real}} : \text{Real} \rightarrow \text{Real}$

# Sequências

$(0, 1, 2, \dots) : \text{Seq Real}$

$(0, 1, 0, 1, \dots) : \text{Seq Real}$

$(x_n)_n \equiv x_0, x_1, x_2, \dots$

$(x_n)_{n \in \mathbb{N}} \quad (x_n)_{n=0}^{\infty}$

$(\underbrace{n^2}_{n \in \mathbb{N}})$

Roubaram o Seq?

$\text{Seq } \alpha \stackrel{\text{impl}}{\equiv} \text{Nat} \rightarrow \alpha$

$\text{Seq} : \text{Type} \rightarrow \text{Type}$

$\text{Set} : \text{Type} \rightarrow \text{Type}$

$\text{Seq (Seq Real)}$

$\text{Set (Seq (Set Real))}$

$\vdots$

Quantas perguntas a um  $A : \text{Set } \alpha$ ?

Tantas quantos os elems de  $\alpha$ .

Quantas perguntas a uma  $(a_n)_n : \text{Seq } \alpha$ ?

“Uma para cada dia.”

↳ Tantas quantos os elems de  $\text{Nat}$

$(x_n)_n \stackrel{\text{Seq } \alpha}{=} (y_n)_n \stackrel{\text{def}}{\iff} (\forall n : \text{Nat}) [x_n = y_n]$

# Exemplos

$$\left(\frac{1}{2^n}\right)_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \dots$$

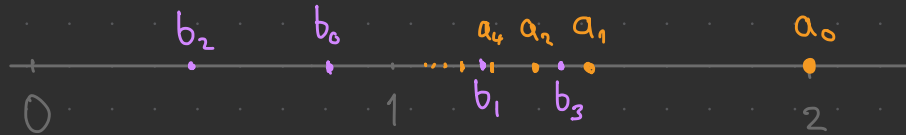
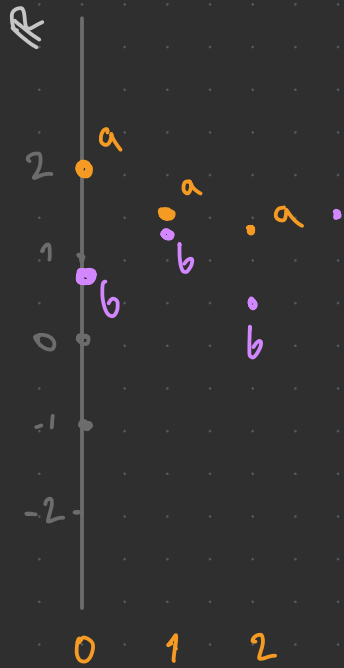
$$(1)_{n \in \mathbb{N}} = 1, 1, 1, \dots$$

$$\left(\frac{n+1}{n}\right)_{n=2}^{\infty} = \frac{3}{2}, \frac{4}{3}, \dots$$

$$b_n \stackrel{\text{def}}{=} 1 + \frac{1}{n} \quad \rightarrow \quad 2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \dots$$

$$\begin{cases} a_0 \stackrel{\text{def}}{=} 0 \\ a_{n+1} \stackrel{\text{def}}{=} 1 - a_n \end{cases} \quad \rightarrow \quad 0, 1, 0, 1, \dots$$

# Desenhando seqüências



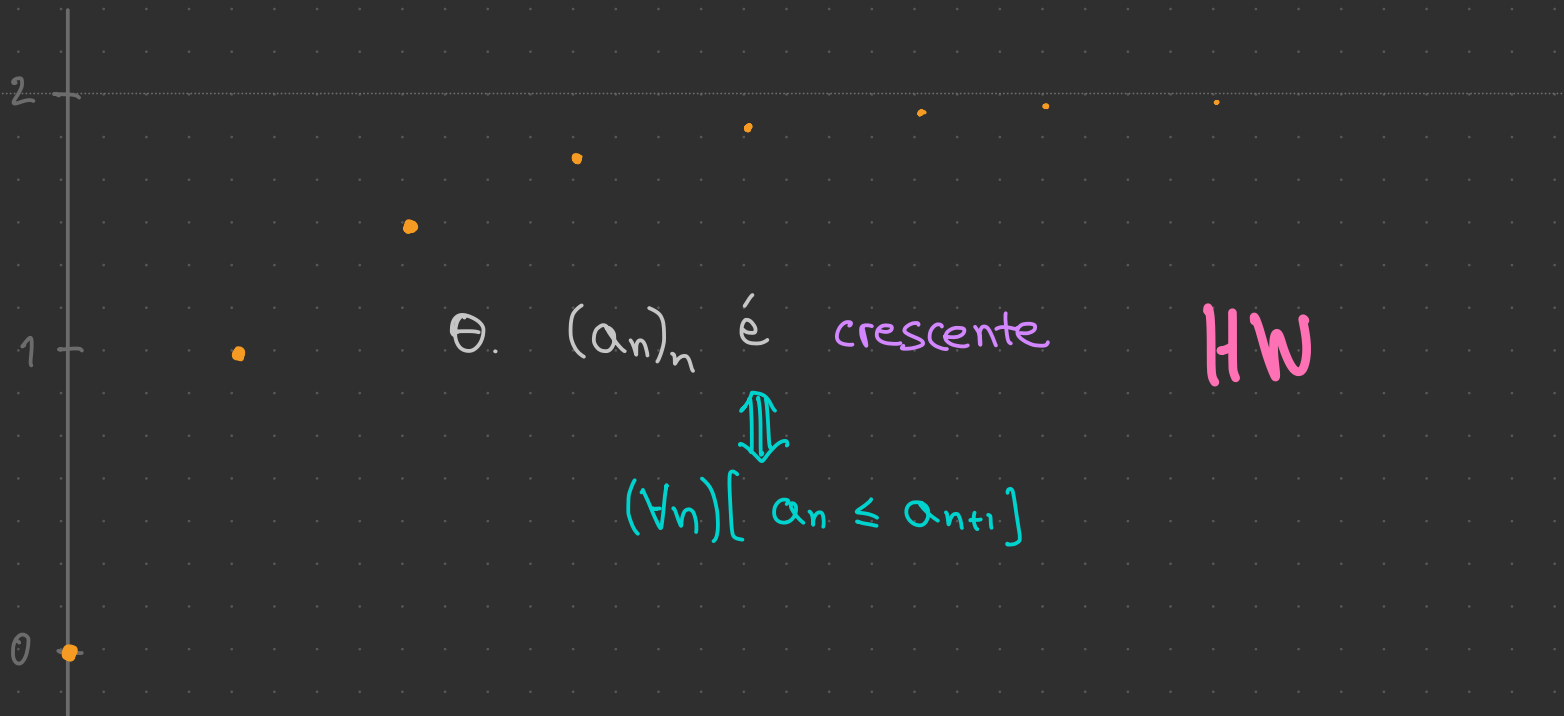


$$\begin{cases} a_0 & \parallel \parallel \parallel 0 \\ a_{n+1} & \parallel \parallel \parallel (1/2) a_n + 1 \end{cases} \quad 0, 1, 1 + \frac{1}{2}, 1 + \frac{3}{4}, \dots$$

$$?, 2 - \frac{1}{1}, 2 - \frac{1}{2}, 2 - \frac{1}{4}, \dots$$

$$2 - \frac{1}{2^3}$$





$\Leftrightarrow (a_n)_n$  é crescente

HW



$$(\forall n) [ a_n \leq a_{n+1} ]$$

# Vocabulário

$(a_n)_n$  (estritamente) crescente  
decrecente  $\overset{\text{def}}{\iff} \dots$

$(a_n)_n$  cotada superiormente  
inferiormente  $\overset{\text{def}}{\iff} (\exists c) [ \text{todos os } a_n \leq c ]$   
 $(\forall n) [ a_n \leq c ]$

Investigar:

$(a_n)_n$  estritamente crescente  $\overset{?}{\implies} (a_n)_n$  não é cot. sup.