FMC1, 2022.2 (Turma N12 do Thanos)

Prova 2.3 (points: 56; bonus: $0^\flat$; time: 48')

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Regras:

I. Não vires esta página antes do começo da prova.

II. Nenhuma consulta de qualquer forma.

III. Nenhum aparelho ligado (por exemplo: celular, tablet, notebook, *etc.*).[1]

IV. Nenhuma comunicação de qualquer forma e para qualquer motivo.

V. $(\forall x)\,[\,\text{Colar}(x) \implies \neg\text{Passar}(x, \text{FMC1})\,]$.[2]

VI. Responda dentro das caixas indicadas, escrevendo em forma clara e facilmente legível.

VII. Nenhuma prova será aceita depois do fim do tempo—mesmo se for atraso de 1 segundo.

VIII. Respeite as restrições dos problemas que têm escolha.[3]

IX. Escolha até um dos L, T.[4]

Esclarecimento: Tuas demonstrações precisam ser escritas na linguagem mid-level que temos elaborado na disciplina.[5] Tuas definições devem utilizar apenas a sintaxe e a notação que temos utilizado na disciplina.

Dados:

```
data Nat           data Maybe α          data List α
  O : Nat            Nothing : Maybe α     Nil  : List α
  S : Nat → Nat      Just    : α → Maybe α Cons : α → List α → List α

data Either α β    data Tree α                      data Dir
  Left  : α → Either α β   Tip  : α → Tree α          L : Dir
  Right : β → Either α β   Fork : Tree α → Tree α → Tree α   R : Dir

(+) : Nat → Nat → Nat   (*) : Nat → Nat → Nat     (++) : List α → List α → List α
m + O     = m           m * O     = O             []     ++ ys = ys
m + (S n) = S (m + n)   m * (S n) = n + (m * n)   (x:xs) ++ ys = x : (xs ++ ys)

(.) : (b → c) → (a → b) → (a → c)
(f . g) x = f (g x)                                type Path = List Dir
```

Os teoremas: (+)-ass/com/id/inv, (·)-ass/com, (≤)-refl/trans/min/succ.

Boas provas!

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[1] Ou seja, *desligue antes* da prova.
[2] Se essa regra não faz sentido, melhor desistir desde já.
[3] Respostas violando essa regra (respondendo em mais questões) tirarão 0 pontos.
[4] Provas violando essa regra (com respostas em mais problemas) não serão corrigidas (tirarão 0 pontos).
[5] *Não inclua* os Dados/Alvo nem outros rascunhos no teu texto!

(38) # L

(12) **L1.** Complete as igualdades seguintes **com algo interessante:**[6]

$$\text{reverse} \circ \text{reverse} = \text{id} \qquad \text{map id} = \text{id}$$
$$\text{map } f \circ \text{map } g = \text{map } (f \circ g) \qquad \text{filter } p \circ \text{map } f = \text{map } f \circ \text{filter } (p \circ f)$$
$$\text{map } f\ (xs +\!\!+ ys) = \text{map } f\ xs +\!\!+ \text{map } f\ ys \qquad \text{sum} \circ \text{map } (n \cdot) = (n \cdot) \circ \text{sum}$$

(6) **L2.** Defina recursivamente as funções: reverse, map, filter.
Definições.

$$\begin{aligned}
&\text{reverse} : \mathsf{List}\ \alpha \to \mathsf{List}\ \alpha \\
&\text{reverse}\ [\,] \quad = [\,] \\
&\text{reverse}\ (x : xs) \quad = \text{reverse}\ xs +\!\!+ [x]
\end{aligned}$$

$$\begin{aligned}
&\text{map} : (\alpha \to \beta) \to \mathsf{List}\ \alpha \to \mathsf{List}\ \beta \\
&\text{map}\ f\ [\,] \quad = [\,] \\
&\text{map}\ f\ (x :: xs) \quad = [\,]
\end{aligned}$$

$$\begin{aligned}
&\text{filter} : (\alpha \to \mathsf{Bool}) \to \mathsf{List}\ \alpha \to \mathsf{List}\ \alpha \\
&\text{filter}\ p\ [\,] \quad = [\,] \\
&\text{filter}\ p\ (x :: xs) \quad = \textbf{if}\ p\ x\ \textbf{then}\ x :: (\text{filter}\ p\ xs)\ \textbf{else}\ \text{filter}\ p\ xs
\end{aligned}$$

(20) **L3.** Escolha **exatamente uma da primeira coluna** do **L1** para demonstrar.
Demonstração da _____ .

$(\text{map } f \circ \text{map } g)\ \ell \stackrel{?}{=} \text{map}\ (f \circ g)\ \ell$.
Por indução no ℓ.
Caso $[\,]$.

$$\begin{aligned}
&(\text{map } f \circ \text{map } g)\ [\,] \\
&= \text{map } f\ (\text{map } g\ [\,]) && ((\circ).1) \\
&= \text{map } f\ [\,] && (\text{map}.1) \\
&= [\,] && (\text{map}.1)
\end{aligned}$$

$$\begin{aligned}
&\text{map}\ (f \circ g)\ [\,] \\
&= [\,] \quad (\text{map}.1)
\end{aligned}$$

Caso $(x :: xs)$.

$$\begin{aligned}
&(\text{map } f \circ \text{map } g)\ (x :: xs) \\
&= \text{map } f\ (\text{map } g\ (x :: xs)) && ((\circ).1) \\
&= \text{map } f\ (g\ x :: \text{map } g\ xs) && (\text{map}.2) \\
&= f\ (g\ x) :: \text{map } f\ (\text{map } g\ xs) && (\text{map}.2) \\
&= (f \circ g)\ x :: \text{map } f\ (\text{map } g\ xs) && ((\circ).1) \\
&= (f \circ g)\ x :: \text{map}\ (f \circ g)\ xs && (\text{HI}) \\
&= \text{map}\ (f \circ g)\ (x :: xs) && (\text{map}.2)
\end{aligned}$$

$(\text{reverse} \circ \text{reverse})\ \ell \stackrel{?}{=} \text{id}\ \ell$.
Calculamos.

$$\begin{aligned}
&(\text{reverse} \circ \text{reverse})\ \ell \\
&= \text{reverse}\ (\text{reverse}\ \ell) && ((\circ).1) \\
&= \ell && ((\text{rev-rev})) \\
&= \text{id}\ \ell && ((\text{id}).1))
\end{aligned}$$

$\text{map } f\ (xs +\!\!+ ys) \stackrel{?}{=} \text{map } f\ xs +\!\!+ \text{map } f\ ys$.
Por indução no xs.
Caso $[\,]$.

$$\begin{aligned}
&\text{map } f\ ([\,] +\!\!+ ys) \\
&= \text{map } f\ ys && ((+\!\!+).1)
\end{aligned}$$

$$\begin{aligned}
&\text{map } f\ [\,] +\!\!+ \text{map } f\ ys \\
&= [\,] +\!\!+ \text{map } f\ ys && (\text{map}.1) \\
&= \text{map } f\ ys && ((+\!\!+).1)
\end{aligned}$$

Caso $(x :: xs)$.

$$\begin{aligned}
&\text{map } f\ ((x : xs) +\!\!+ ys) \\
&= \text{map } f\ (x : (xs +\!\!+ ys)) && ((+\!\!+).2) \\
&= f\ x : \text{map } f\ (xs +\!\!+ ys) && (\text{map}.2) \\
&= f\ x : (\text{map } f\ xs +\!\!+ \text{map } f\ ys) && (\text{HI}) \\
&= (f\ x : \text{map } f\ xs) +\!\!+ \text{map } f\ ys && ((+\!\!+).2) \\
&= \text{map } f\ (x : xs) +\!\!+ \text{map } f\ ys && (\text{map}.2)
\end{aligned}$$

[6] Definição. Chamamos algo de *interessante* sse Thanos acha tal algo interessante.

(56) # T

(7) **T1.** Escreva a regra de inferência que corresponde à indução do tipo Tree α.

$$\frac{(\forall x : \alpha)\,[\,\varphi(\mathsf{Tip}\ x)\,] \qquad (\forall l, r : \mathsf{Tree}\ \alpha)\,[\,\varphi(l)\ \&\ \varphi(r) \implies \varphi(\mathsf{Fork}\ l\ r)\,]}{(\forall t : \mathsf{Tree}\ \alpha)\,[\,\varphi(t)\,]}\ \textsc{Ind}_{\varphi}^{\mathsf{Tree}\ \alpha}$$

(5) **T2.** Escreva **apenas nomes e tipagens** para os destrutores do tipo Tree α.[7]

$$\mathrm{tip} : \mathsf{Tree}\ \alpha \to \alpha \qquad\qquad \mathrm{lchild}, \mathrm{rchild} : \mathsf{Tree}\ \alpha \to \mathsf{Tree}\ \alpha$$

(4) **T3.** Defina o que precisa para o Tree virar um Functor.[8]
Definição.

$$\begin{aligned} &\mathrm{fmap} : (\alpha \to \beta) \to \mathsf{Tree}\ \alpha \to \mathsf{Tree}\ \beta \\ &\mathrm{fmap}\ f\ (\mathsf{Tip}\ x) \ = \mathsf{Tip}\ (f\ x) \\ &\mathrm{fmap}\ f\ (\mathsf{Fork}\ l\ r) \ = \mathsf{Fork}\ (\mathrm{fmap}\ f\ l)\ (\mathrm{fmap}\ f\ r) \end{aligned}$$

(16) **T4.** Levando em consideração os exemplos de uso no quadro, defina recursivamente as funções:

(2×) $\mathrm{forks}, \mathrm{tips} : \mathsf{Tree}\ \alpha \to \mathsf{Nat}$ $\qquad$ $\mathrm{flat} : \mathsf{Tree}\ \alpha \to \mathsf{List}\ \alpha$

(5×) $\mathrm{search} : \alpha \to \mathsf{Tree}\ \alpha \to \mathsf{List\ Path}$ $\qquad$ $\mathrm{fetch} : \mathsf{Path} \to \mathsf{Tree}\ \alpha \to \mathsf{Maybe}\ \alpha.$

Resposta. **Não** repita as tipagens na resposta!

```
forks (Tip _)    = O
forks (Fork l r) = S (forks l + forks r)

tips (Tip _)    = S O
tips (Fork l r) = tips l + tips r

flat (Tip x)    = [x]
flat (Fork l r) = flat (l) ++ flat (r)

search w (Tip x)    = if w = x then [[]] else []
search w (Fork l r) = let (lpaths, rpaths) = (search w l, search w r)
                      in  map (L ::) lpaths ++ map (R ::) rpaths

fetch (L :: ds) (Fork l _) = fetch ds l
fetch (R :: ds) (Fork _ r) = fetch ds r
fetch []        (Tip x)    = Just x
fetch _         _          = Nothing
```

[7] Versões parciais/unsafe!
[8] **Com** tipagem; e **sem** demonstrar as leis necessárias!

(24) **T5.** Demonstre **exatamente uma das:**

(16) (i) $\mathit{tips} = \mathsf{S} \circ \mathit{forks}$;

(24) (ii) $\mathit{flat} \circ \mathit{fmap}\ f = \mathit{map}\ f \circ \mathit{flat}$.

Podes considerar dados quaisquer dos teoremas da **L1.**

Demonstração.

(i) $\mathit{tips}\ t \stackrel{?}{=} (\mathsf{S} \circ \mathit{forks})\ t$.
Por indução na t.

Caso $(\mathsf{Tip}\ x)$.

$$
\begin{array}{ll}
(\mathsf{S} \circ \mathit{forks})\ (\mathsf{Tip}\ x) & \\
\quad = \mathsf{S}\ (\mathit{forks}\ (\mathsf{Tip}\ x)) & ((\circ).1) \\
\quad = \mathsf{S}\ \mathsf{O} & ((\mathit{forks}).1) \\
\quad = \mathit{tips}\ (\mathsf{Tip}\ x) & ((\mathit{tips}).1)
\end{array}
$$

Caso $(\mathsf{Fork}\ l\ r)$.

$$
\begin{array}{ll}
\mathit{tips}\ (\mathsf{Fork}\ l\ r) & \\
\quad = \mathit{tips}\ l + \mathit{tips}\ r & ((\mathit{tips}).2) \\
\quad = (\mathsf{S} \circ \mathit{forks})\ l + (\mathsf{S} \circ \mathit{forks})\ r & (\mathrm{HI}) \\
\quad = \mathsf{S}\ (\mathit{forks}\ l) + \mathsf{S}\ (\mathit{forks}\ r) & ((\circ).1) \\
\quad = \mathsf{S}\ (\mathsf{S}\ (\mathit{forks}\ l) + \mathit{forks}\ r) & ((+).2) \\
\quad = \mathsf{S}\ (\mathit{forks}\ r + \mathsf{S}\ (\mathit{forks}\ l)) & ((+)\text{-com.}) \\
\quad = \mathsf{S}\ (\mathsf{S}\ (\mathit{forks}\ r + \mathit{forks}\ l)) & ((+).2) \\
\quad = \mathsf{S}\ (\mathsf{S}\ (\mathit{forks}\ l + \mathit{forks}\ r)) & ((+)\text{-com.}) \\
\quad = \mathsf{S}\ (\mathit{forks}\ (\mathsf{Fork}\ l\ r)) & (\mathit{forks}.2) \\
\quad = (\mathsf{S} \circ \mathit{forks})\ (\mathsf{Fork}\ l\ r) & ((\circ).1)
\end{array}
$$

(ii) $(\mathit{flat} \circ \mathit{fmap}\ f)\ t \stackrel{?}{=} (\mathit{map}\ f \circ \mathit{flat})\ t$.
Por indução na t.

Caso $(\mathsf{Tip}\ x)$.

$$
\begin{array}{ll}
(\mathit{flat} \circ \mathit{fmap}\ f)\ (\mathsf{Tip}\ x) & \\
\quad = \mathit{flat}\ (\mathit{fmap}\ f\ (\mathsf{Tip}\ x)) & ((\circ).1) \\
\quad = \mathit{flat}\ (\mathsf{Tip}\ (f\ x)) & (\mathit{fmap}.1) \\
\quad = [f\ x] & (\mathit{flat}.1) \\
 & \\
(\mathit{map}\ f \circ \mathit{flat})\ (\mathsf{Tip}\ x) & \\
\quad = \mathit{map}\ f\ (\mathit{flat}\ (\mathsf{Tip}\ x)) & ((\circ).1) \\
\quad = \mathit{map}\ f\ ([x]) & (\mathit{flat}.1) \\
\quad \stackrel{\text{sug}}{\equiv} \mathit{map}\ f\ (x :: [\,]) & \\
\quad = f\ x :: \mathit{map}\ f\ [\,] & (\mathit{map}.2) \\
\quad = f\ x :: [\,] & (\mathit{map}.1) \\
\quad \stackrel{\text{sug}}{\equiv} [f\ x] &
\end{array}
$$

Caso $(\mathsf{Fork}\ l\ r)$.

$$
\begin{array}{ll}
(\mathit{flat} \circ \mathit{fmap}\ f)\ (\mathsf{Fork}\ l\ r) & \\
\quad = \mathit{flat}\ (\mathit{fmap}\ f\ (\mathsf{Fork}\ l\ r)) & ((\circ).1) \\
\quad = \mathit{flat}\ (\mathsf{Fork}\ (\mathit{fmap}\ f\ l)\ (\mathit{fmap}\ f\ r)) & (\mathit{fmap}.2) \\
\quad = \mathit{flat}\ (\mathit{fmap}\ f\ l) +\!\!+ \mathit{flat}\ (\mathit{fmap}\ f\ r) & (\mathit{flat}.2) \\
\quad = (\mathit{flat} \circ \mathit{fmap}\ f)\ l +\!\!+ (\mathit{flat} \circ \mathit{fmap}\ f)\ r & ((\circ).1) \\
\quad = (\mathit{map}\ f \circ \mathit{flat})\ l +\!\!+ (\mathit{flat} \circ \mathit{fmap}\ f)\ r & (\mathrm{HI}\text{ de }(l)) \\
\quad = (\mathit{map}\ f \circ \mathit{flat})\ l +\!\!+ (\mathit{map}\ f \circ \mathit{flat})\ r & (\mathrm{HI}\text{ de }(r)) \\
\quad = \mathit{map}\ f\ (\mathit{flat}\ l) +\!\!+ \mathit{map}\ f\ (\mathit{flat}\ r) & ((\circ).1) \\
\quad = \mathit{map}\ f\ (\mathit{flat}\ l +\!\!+ \mathit{flat}\ r) & (\text{map-distr-}(+\!\!+)) \\
\quad = \mathit{map}\ f\ (\mathit{flat}\ (\mathsf{Fork}\ l\ r)) & (\mathit{flat}.2) \\
\quad = (\mathit{map}\ f \circ \mathit{flat})\ (\mathsf{Fork}\ l\ r) & ((\circ).1)
\end{array}
$$

Obs: o (map-distr-(++)) está na **L1.**

Só isso mesmo.

rev-rev.
$\text{rev}\ (\text{rev}\ \ell) = \ell$

DEMONSTRAÇÃO.
Por indução na ℓ.
CASO $[\,]$:

$$\begin{aligned}
&\text{rev}\ (\text{rev}\ [\,]) \\
&\quad = \text{rev}\ [\,] \\
&\quad = [\,]
\end{aligned}$$

CASO $(x :: xs)$:

$$\begin{aligned}
&\text{rev}\ (\text{rev}\ ((\,)\, x :: xs)) \\
&\quad = \text{rev}\ (\text{rev} xs +\!\!+ [x]) && (\text{rev}.2) \\
&\quad = \text{rev}\ [x] +\!\!+ \text{rev}\ (\text{rev}\ xs) && ((\text{rev-}(+\!\!+))) \\
&\quad = \text{rev}\ [x] +\!\!+ xs && (\text{HI}) \\
&\quad \overset{\text{sug}}{\equiv} \text{rev}\ (x :: [\,]) +\!\!+ xs \\
&\quad = (\text{rev}\ [\,] +\!\!+ [x]) +\!\!+ xs && (\text{rev}.2) \\
&\quad = ([\,] +\!\!+ [x]) +\!\!+ xs && (\text{rev}.1) \\
&\quad = [x] +\!\!+ xs && ((+\!\!+).1)
\end{aligned}$$

rev-(++).
$\text{rev}\ (xs +\!\!+ ys) = \text{rev}\ ys +\!\!+ \text{rev}\ xs.$

DEMONSTRAÇÃO.
Por indução na xs.
CASO $[\,]$:

$$\begin{aligned}
&\text{rev}\ ([\,] +\!\!+ ys) \\
&\quad = \text{rev}\ ys \qquad ((+\!\!+).1) \\
&\text{rev}\ ys +\!\!+ \text{rev}\ [\,] \\
&\quad = \text{rev}\ ys +\!\!+ [\,] \qquad (\text{rev}\ .1) \\
&\quad = \text{rev}\ ys \qquad ((\text{nil-idR}))
\end{aligned}$$

CASO $(x :: xs)$:

$$\begin{aligned}
&\text{rev}\ ((x :: xs) +\!\!+ ys) \\
&\quad = \text{rev}\ (x :: (xs +\!\!+ ys)) \\
&\quad = \text{rev}\ (xs +\!\!+ ys) +\!\!+ [x] && (\text{rev}.2) \\
&\quad = (\text{rev}\ ys +\!\!+ \text{rev}\ xs) +\!\!+ [x] && (\text{HI}) \\
&\quad = \text{rev}\ ys +\!\!+ (\text{rev}\ xs +\!\!+ [x]) && ((+\!\!+)\text{-ass.}) \\
&\quad = \text{rev}\ ys +\!\!+ (\text{rev}\ (x :: xs)) && ((+\!\!+)\text{-ass.})
\end{aligned}$$

nil-idR.
$\ell +\!\!+ [\,] = \ell$

DEMONSTRAÇÃO.
Por indução na ℓ.
CASO $[\,]$: $[\,] +\!\!+ [\,] = [\,]$.
CASO $(x :: xs)$: $(x :: xs) +\!\!+ [\,] = x :: (xs +\!\!+ [\,]) = x :: xs$.

(++)-ass.
$(xs +\!\!+ ys) +\!\!+ zs = xs +\!\!+ (ys +\!\!+ zs)$

DEMONSTRAÇÃO.
Por indução na xs.
CASO $[\,]$:

$$\begin{aligned}
&[\,] +\!\!+ (ys +\!\!+ zs) \\
&\quad = ys +\!\!+ zs \\
&([\,] +\!\!+ ys) +\!\!+ zs \\
&\quad = ys +\!\!+ zs
\end{aligned}$$

CASO $(x :: xs)$:

$$\begin{aligned}
&(x :: xs) +\!\!+ (ys +\!\!+ zs) \\
&\quad = x :: (xs +\!\!+ (ys +\!\!+ zs)) && (+\!\!+.2) \\
&\quad = x :: ((xs +\!\!+ ys) +\!\!+ zs) && (\text{HI}) \\
&\quad = (x :: (xs +\!\!+ ys)) +\!\!+ zs && (+\!\!+.2) \\
&\quad = ((x :: xs) +\!\!+ ys) +\!\!+ zs && (+\!\!+.2)
\end{aligned}$$